ON EXTREMELY AMENABLE GROUPS OF HOMEOMORPHISMS

by

VLADIMIR USPENSKIJ

Electronically published on August 5, 2008
ON EXTREMELY AMENABLE GROUPS OF HOMEOMORPHISMS

VLADIMIR USPENSKIJ

ABSTRACT. A topological group $G$ is extremely amenable if every compact $G$-space has a $G$-fixed point. Let $X$ be compact and $G \subset \text{Homeo}(X)$. We prove that the following are equivalent: (1) $G$ is extremely amenable; (2) every minimal closed $G$-invariant subset of $\text{Exp} R$ is a singleton, where $R$ is the closure of the set of all graphs of $g \in G$ in the space $\text{Exp} \; (X^2)$ ($\text{Exp}$ stands for the space of closed subsets); (3) for each $n = 1, 2, \ldots$ there is a closed $G$-invariant subset $Y_n$ of $(\text{Exp} \; X)^n$ such that $\bigcup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of $X$ and for every $n \geq 1$ every minimal closed $G$-invariant subset of $\text{Exp} Y_n$ is a singleton. This yields an alternative proof of Pestov’s theorem that the group of all order-preserving self-homeomorphisms of the Cantor middle-third set (or of the interval $[0, 1]$) is extremely amenable.

1. Introduction

With every\(^1\) topological group $G$ one can associate the greatest ambit $S(G)$ and the universal minimal compact $G$-space $\mathcal{M}(G)$. To define these objects, recall some definitions. A $G$-space is a topological space $X$ with a continuous action of $G$, that is, a map

\(^1\)All spaces are assumed to be Tikhonov, and all maps are assumed to be continuous.
\( G \times X \to X \) satisfying \( g(hx) = (gh)x \) and \( 1x = x \) \((g, h \in G, x \in X)\). A map \( f : X \to Y \) between two \( G \)-spaces is \( G \)-equivariant, or a \( G \)-map for short, if \( f(gx) = gf(x) \) for every \( g \in G \) and \( x \in X \).

A semigroup is a set with an associative multiplication. A semigroup \( X \) is right topological if it is a topological space and for every \( y \in X \) the self-map \( x \mapsto xy \) of \( X \) is continuous. (Sometimes the term left topological is used for the same thing.) A subset \( I \subset X \) is a left ideal if \( XI \subset I \). If \( G \) is a topological group, a right topological semigroup compactification of \( G \) is a right topological compact semigroup \( X \) together with a continuous semigroup morphism \( f : G \to X \) with a dense range such that the map \((g, x) \mapsto f(g)x\) from \( G \times X \) to \( X \) is jointly continuous (and hence \( X \) is a \( G \)-space).

The greatest ambit \( S(G) \) for \( G \) is a right topological semigroup compactification which is universal in the usual sense: for any right topological semigroup compactification \( X \) of \( G \) there is a unique morphism \( S(G) \to X \) of right topological semigroups such that the obvious diagram commutes. Considered as a \( G \)-space, \( S(G) \) is characterized by the following property: there is a distinguished point \( e \in S(G) \) such that for every compact \( G \)-space \( Y \) and every \( a \in Y \) there exists a unique \( G \)-map \( f : S(G) \to Y \) such that \( f(e) = a \).

We can take for \( S(G) \) the compactification of \( G \) corresponding to the \( C^* \)-algebra \( \text{RUCB}(G) \) of all bounded right uniformly continuous functions on \( G \), that is, the maximal ideal space of that algebra. (A complex function \( f \) on \( G \) is right uniformly continuous if

\[
\forall \epsilon > 0 \exists V \in \mathcal{N}(G) \forall x, y \in G \ (xy^{-1} \in V \Rightarrow |f(y) - f(x)| < \epsilon),
\]

where \( \mathcal{N}(G) \) is the filter of neighbourhoods of unity.) The \( G \)-space structure on \( S(G) \) comes from the natural continuous action of \( G \) by automorphisms on \( \text{RUCB}(G) \) defined by \( gf(h) = f(g^{-1}h) \) \((g, h \in G, f \in \text{RUCB}(G))\). We shall identify \( G \) with a subspace of \( S(G) \). Closed \( G \)-subspaces of \( S(G) \) are the same as closed left ideals of \( S(G) \).

A \( G \)-space \( X \) is minimal if it has no proper \( G \)-invariant closed subsets or, equivalently, if the orbit \( Gx \) is dense in \( X \) for every \( x \in X \). The universal minimal compact \( G \)-space \( M(G) \) is characterized by the following property: \( M(G) \) is a minimal compact \( G \)-space, and for every compact minimal \( G \)-space \( X \) there exists a \( G \)-map of \( M(G) \) onto \( X \). Since Zorn’s lemma implies that every compact \( G \)-space has a minimal compact \( G \)-subspace, it follows that for every
compact $G$-space $X$, minimal or not, there exist a $G$-map of $\mathcal{M}(G)$ to $X$. The space $\mathcal{M}(G)$ is unique up to a $G$-space isomorphism and is isomorphic to any minimal closed left ideal of $\mathcal{S}(G)$, see e.g. [1], [9, Section 4.1], [11, Appendix], [10, Theorem 3.5].

A topological group $G$ is extremely amenable if $\mathcal{M}(G)$ is a singleton or, equivalently, if $G$ has the fixed point on compacta property: every compact $G$-space $X$ has a $G$-fixed point, that is, a point $p \in X$ such that $gp = p$ for every $g \in G$. Examples of extremely amenable groups include $\text{Homeo}_+[0,1] = \text{the group of all orientation-preserving self-homeomorphisms of } [0,1]$; $U_\mathcal{S}(H) = \text{the unitary group of a Hilbert space } H$, with the topology inherited from the product $H^H$; $\text{Iso}(U) = \text{the group of isometries of the Urysohn universal metric space } U$. See Pestov’s book [9] for the proof. Note that a locally compact group $\neq \{1\}$ cannot be extremely amenable, since every locally compact group admits a free action on a compact space [12], [9, Theorem 3.3.2].

We refer the reader to Pestov’s book [9] for various intrinsic characterizations of extremely amenable groups. These characterizations reveal a close connection between Ramsey theory and the notion of extreme amenability. The aim of the present paper is to give another characterization of extremely amenable groups, based on a different approach. For a compact space $X$ let $H(X)$ be the group of all self-homeomorphisms of $X$, equipped with the compact-open topology. Let $G$ be a topological subgroup of $H(X)$. There is an obvious necessary condition for $G$ to be extremely amenable: every minimal closed $G$-subset of $X$ must be a singleton. However, this condition is not sufficient. For example, let $X$ be the Hilbert cube, and let $G \subseteq H(X)$ be the stabilizer of a given point $p \in X$. Then the only minimal closed $G$-subset of $X$ is the singleton $\{p\}$, but $G$ is not extremely amenable [11], since $G$ acts without fixed points on the compact space $\Phi_p$ of all maximal chains of closed subsets of $X$ starting at $p$. The space $\Phi_p$ is a subspace of the compact $G$-space $\text{Exp} \text{Exp} X$, where for a compact space $K$ we denote by $\text{Exp} K$ the compact space of all closed non-empty subsets of $K$, equipped with the Vietoris topology\footnote{If $F$ is closed in $K$, the sets $\{A \in \text{Exp} K : A \subseteq F\}$ and $\{A \in \text{Exp} K : A$ meets $F\}$ are closed in $\text{Exp} K$, and the Vietoris topology is generated by the closed sets of this form. If $K$ is a $G$-space, then so is $\text{Exp} K$, in an obvious way.}. It was indeed necessary to use the second exponent in this example, the first exponent
would not work. One can ask whether in general for every group $G \subset H(X)$ which is not extremely amenable there exists a compact $G$-space $X'$ derived from $X$ by applying a small number of simple functors, like powers, probability measures, exponents, etc., such that $X'$ contains a closed $G$-subspace (which can be taken minimal) on which $G$ acts without fixed points. We answer this question in the affirmative.

Consider the action of $G$ on $\text{Exp}(X^2)$ defined by the composition of relations: if $g \in G$, $F \subset X^2$, and $\Gamma_g \subset X^2$ is the graph of $g$, then $gF = \Gamma_g \circ F = \{(x, gy): (x, y) \in F\}$. This amounts to considering $X^2$ as the product of two different $G$-spaces: the first copy of $X$ has the trivial $G$-structure, and the second copy is the given $G$-space $X$. If $G$ is not extremely amenable, then there is a closed minimal $G$-subspace $Y$ of $\text{Exp}\text{Exp}(X^2)$ that is not a singleton (and hence fixed point free). This follows from:

**Theorem 1.1.** Let $X$ be compact, $G$ a subgroup of $H(X)$. Denote by $R$ the closure of the set $\{\Gamma_g: g \in G\}$ of the graphs of all $g \in G$ in the space $\text{Exp}(X^2)$. Then $G$ is extremely amenable if and only if every minimal closed $G$-subset of $\text{Exp} R$ is a singleton.

Here $X^2$ is the product of the trivial $G$-space and the given $G$-space $X$, as in the paragraph preceding Theorem 1.1, and $R$ is considered as a $G$-subspace of $\text{Exp}(X^2)$.

For example, let $X = I = [0, 1]$ be the closed unit interval. Consider the group $G = H_+([0, 1])$ of all orientation-preserving self-homeomorphisms of $I$. The space $R$ in this case consists of all curves $\Gamma$ in the square $I^2$ that connect the lower left and upper right corners and “never go down”: if $(x, y) \in \Gamma$, $(x', y') \in \Gamma$ and $x < x'$, then $y \leq y'$ (see the picture in [8, Example 2.5.4]). It can be verified that the only minimal compact $G$-subsets of $\text{Exp} R$ are singletons (they are of the form $\{a \text{ union of } G\text{-orbits in } R\}$). The proof depends on the following lemma:

**Lemma 1.2.** Let $\Delta^n$ be the $n$-simplex of all $n$-tuples $(x_1, \ldots, x_n) \in I^n$ such that $0 \leq x_1 \leq \cdots \leq x_n \leq 1$. Equip $\Delta^n$ with the natural action of the group $G = H_+([0, 1])$. Then every minimal closed $G$-subset of $\text{Exp} \Delta^n$ is a singleton ($= \{a \text{ union of some faces of } \Delta^n\}$).
The idea to consider the action of $G = H_+([0, 1])$ on $\Delta^n$ is borrowed from [2], where it is shown that the geometric realization of any simplicial set can be equipped with a natural action of $G$. We shall not prove Lemma 1.2, since this lemma follows from Pestov's theorem that $G$ is extremely amenable, and I am not aware of a short independent proof of the lemma. The essence of the lemma is that every subset of $\Delta^n$ can be either pushed (by an element of $G$) into the $\epsilon$-neighbourhood of the boundary of the simplex or else can be pushed to approximate the entire simplex within $\epsilon$. Some Ramsey-type argument seems to be necessary for this. Actually Lemma 1.2 may be viewed as a topological equivalent of the finite Ramsey theorem [9, Theorem 1.5.2], since Pestov showed that this theorem has an equivalent reformulation in terms of the notion of a “finitely oscillation stable” dynamical system [9, Section 1.5], and extremely amenable groups are characterized in the same terms [9, Theorem 2.1.11].

An important example of an extremely amenable group is the Polish group $\text{Aut}(\mathbb{Q})$ of all automorphisms of the ordered set $\mathbb{Q}$ of rationals [6], [9, Theorem 2.3.1]. This group is considered with the topology inherited from $(\mathbb{Q}_d)^\mathbb{Q}$, where $\mathbb{Q}_d$ is the set of rationals with the discrete topology. Let $K \subset [0, 1]$ be the usual middle-third Cantor set. The topological group $\text{Aut}(\mathbb{Q})$ is isomorphic to the topological group $G = H_<(K) \subset H(K)$ of all order-preserving self-homeomorphisms of $K$. To see this, note that pairs of the endpoints of “deleted intervals” (= components of $[0, 1] \setminus K$) form a set which is order-isomorphic to $\mathbb{Q}$, whence a homomorphism $G \to \text{Aut}(\mathbb{Q})$ which is easily verified to be a topological isomorphism. One can prove that the group $G \simeq \text{Aut}(\mathbb{Q})$ is extremely amenable with the aid of Theorem 1.1. The proof is essentially the same as in the case of the group $G = H_+([0, 1])$. The space $R$ considered in Theorem 1.1 again is the space of “curves”, this time in $K^2$, that go from $(0, 0)$ to $(1, 1)$ and “look like graphs”, with the exception that they may contain vertical and horizontal parts. The evident analogue of Lemma 1.2 holds for “Cantor simplices” of the form $\{(x_1, \ldots, x_n) \in K^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$.

Theorem 1.1 may help to answer the following:

**Question 1.3.** Let $P$ be pseudoarc, $G = H(P)$, and let $G_0$ be the stabilizer of a given point $x \in P$. Is $G_0$ extremely amenable?
As explained in [11], this question is motivated by the observation that the argument involving maximal chains, which shows that the stabilizer \( G_0 \subset H(X) \) of a point \( p \in X \) is not extremely amenable if \( X \) is either a Hilbert cube or a compact manifold of dimension \( >1 \), does not work for the pseudoarc. A positive answer to Question 1.3 would imply that the pseudoarc \( P \) can be identified with \( M(G) \) for \( G = H(P) \). The problem whether this is the case was raised in [11] and appears as Problem 6.7.20 in [9].

The suspension \( \Sigma X \) of a space \( X \) is the quotient of \( X \times I \) obtained by collapsing the “bottom” \( X \times \{0\} \) and the “top” \( X \times \{1\} \) to points. Let \( q : \Sigma X \to I \) be the natural projection. The inverse image under \( q \) of the maximal chain \( \{[0,x] : x \in I\} \) of closed subsets of \( I \) is a maximal chain of closed subsets of \( \Sigma X \).

**Question 1.4.** Let \( Q = I^\omega \) be the Hilbert cube, and \( C \) be the maximal chain of subcontinua of \( \Sigma Q \) considered above. If \( G = H(\Sigma Q) \) and \( G_0 \subset G \) is the stabilizer of \( C \), is \( G_0 \) extremely amenable?

This question is motivated by the search for a good candidate for the space \( M(G) \), where \( G = H(Q) \). The space \( \Phi_c \) of all maximal chains of subcontinua of \( Q \), proved to be minimal by Y. Gutman [5], may be such a candidate [9, Problem 6.4.13]. Recall that for the group \( G = H(K) \), where \( K = 2^\omega \) is the Cantor set, \( M(G) \) can be identified with the space \( \Phi \subset \text{Exp}\text{Exp}K \) of all maximal chains of closed subsets of \( K \) [4], [9, Example 6.7.18].

There is another characterization (Theorem 1.5) of extremely amenable groups in the spirit of Theorem 1.1 which, in combination with Lemma 1.2, readily implies Pestov’s results that \( H_+([0,1]) \) and \( \text{Aut}(\mathbb{Q}) \) are extremely amenable. Let \( X \) be compact, \( Y_n \subset (\text{Exp}X)^n \) for \( n = 1, 2, \ldots \). We say that \( \cup_{n=1}^\infty Y_n \) contains arbitrarily fine covers if for every open cover \( \alpha \) of \( X \) there are \( n \ge 1 \) and \( (F_1, \ldots, F_n) \in Y_n \) such that \( \cup_{i=1}^n F_i = X \) and the cover \( \{F_i\}_{i=1}^n \) of \( X \) refines \( \alpha \).

**Theorem 1.5.** Let \( X \) be compact, \( G \) a subgroup of \( H(X) \). Let \( Y_n \) be a closed \( G \)-invariant subset of \( (\text{Exp}X)^n \) \( (n = 1, 2, \ldots) \) such that \( \cup_{n=1}^\infty Y_n \) contains arbitrarily fine covers of \( X \). Then \( G \) is extremely amenable if and only if for every \( n \ge 1 \) every minimal closed \( G \)-invariant subset of \( \text{Exp}Y_n \) is a singleton.
Observe that Pestov’s theorem asserting that \( G = H_\alpha([0,1]) \) is extremely amenable follows from Theorem 1.5 and Lemma 1.2: it suffices to take for \( Y_{n+1} \) the collection of all sequences 
\[
([0, x_1], [x_1, x_2], \ldots, [x_n, 1]),
\]
where \( 0 \leq x_1 \leq \cdots \leq x_n \leq 1 \). The \( G \)-space \( Y_{n+1} \) is isomorphic to the \( n \)-simplex \( \Delta^n \) considered in Lemma 1.2. The argument for \( \text{Aut} (\mathbb{Q}) \simeq H_\alpha(K) \) is similar.

The proof of Theorems 1.1 and 1.5 depends on the notion of a representative family of compact \( G \)-spaces. We introduce this notion in Section 2 and observe that a topological group \( G \) is extremely amenable if (and only if) there exists a representative family \( \{X_\alpha\} \) such that any minimal closed \( G \)-subset of any \( X_\alpha \) is a singleton (Theorem 2.2). In Section 3 we prove that the single space \( \text{Exp} \mathbb{R} \) considered in Theorem 1.1 constitutes a representative family (Theorem 3.1). The conjunction of Theorems 2.2 and 3.1 proves Theorem 1.1. In Section 4 we prove that under the conditions of Theorem 1.5 the sequence \( \{\text{Exp} Y_n\} \) is representative (Theorem 4.2). The conjunction of Theorems 2.2 and 4.2 proves Theorem 1.5.

2. Representative families of \( G \)-spaces

Let \( G \) be a topological group, \( X \) a compact \( G \)-space. For \( g \in G \) the \( g \)-translation of \( X \) is the map \( x \mapsto gx, x \in X \). The enveloping semigroup (or the Ellis semigroup) \( E(X) \) of the dynamical system \( (G, X) \) is the closure of the set of all \( g \)-translations, \( g \in G \), in the compact space \( X^X \). This is a right topological semigroup compactification of \( G \), as defined in Section 1. The natural map \( G \to E(X) \) extends to a \( G \)-map \( S(G) \to E(X) \) which is a morphism of right topological semigroups.

**Definition 2.1.** A family \( \{X_\alpha : \alpha \in A\} \) of compact \( G \)-spaces is **representative** if the family of natural maps \( S(G) \to E(X_\alpha), \alpha \in A \), separates points of \( S(G) \) (and hence yields an embedding of \( S(G) \) into \( \prod_{\alpha \in A} E(X_\alpha) \)).

**Theorem 2.2.** Let \( G \) be a topological group, \( \{X_\alpha\} \) a representative family of compact \( G \)-spaces. Then \( G \) is extremely amenable if (and only if) every minimal closed \( G \)-subset of every \( X_\alpha \) is a singleton.

This is a special case of a more general theorem:
Theorem 2.3. If \( \{X_\alpha\} \) is a representative family of compact \( G \)-spaces, the universal minimal compact \( G \)-space \( M(G) \) is isomorphic (as a \( G \)-space) to a \( G \)-subspace of a product \( \prod Y_\beta \), where each \( Y_\beta \) is a minimal compact \( G \)-space isomorphic to a \( G \)-subspace of some \( X_\alpha \).

Proof. By definition of a representative family, the greatest ambit \( S(G) \) can be embedded (as a \( G \)-space) into the product \( \prod E(X_\alpha) \) and hence also into the product \( \prod X_\alpha^{X_\alpha} \). Consider \( M(G) \) as a subspace of \( S(G) \) and take for the \( Y_\beta \)'s the projections of \( M(G) \) to the factors \( X_\alpha \). \( \square \)

We now give a sufficient condition for a family of compact \( G \)-spaces to be representative. Let us say that two subsets \( A, B \) of \( G \) are far from each other with respect to the right uniformity if one of the following equivalent conditions holds: (1) the neutral element \( 1_G \) of \( G \) is not in the closure of the set \( BA^{-1} \); (2) for some neighbourhood \( U \) of \( 1_G \) the sets \( A \) and \( UB \) are disjoint; (3) there exists a right uniformly continuous function \( f : G \to [0, 1] \) such that \( f = 0 \) on \( A \) and \( f = 1 \) on \( B \); (4) \( A \) and \( B \) have disjoint closures in \( S(G) \).

Proposition 2.4. Let \( F \) be a family of compact \( G \)-spaces. Suppose that the following holds:

\((*)\) if \( A, B \subset G \) are far from each other with respect to the right uniformity, then there exists \( X \in F \) and \( p \in X \) such that the sets \( Ap \) and \( Bp \) have disjoint closures in \( X \).

Then \( F \) is representative.

Proof. Consider the natural map \( G \to \prod \{E(X) : X \in F\} \). It defines a compactification \( bG \) of \( G \). We must prove that this compactification is equivalent to \( S(G) \).

Let \( A, B \) be any two subsets of \( G \) with disjoint closures in \( S(G) \). Then \( A \) and \( B \) are far from each other with respect to the right uniformity. According to the condition \((*)\), there exists \( X \in F \) and \( p \in X \) such that the sets \( Ap \) and \( Bp \) have disjoint closures in \( X \). It follows that the images of \( A \) and \( B \) in \( E(X) \) have disjoint closures, and \textit{a fortiori} the images of \( A \) and \( B \) in \( bG \) have disjoint closures. It follows that \( S(G) \) and \( bG \) are equivalent compactifications of \( G \) \[3, Theorem 3.5.5\]. \( \square \)
3. Proof of Theorem 1.1

Recall the setting of Theorem 1.1: $X$ is compact, $G$ is a topological subgroup of $H(X)$. For $g \in G$ let $\Gamma_g = \{(x, gx) : x \in X\} \subset X^2$ be the graph of $g$, and let $R$ be the closure of the set $\{\Gamma_g : g \in G\}$ in the compact space $\text{Exp}(X^2)$. We consider the action of $G$ on $\text{Exp}(X^2)$ defined by $gF = \{(x, gy) : (x, y) \in F\}$ ($g \in G$, $F \in \text{Exp}(X^2)$), and consider $R$ as a $G$-subspace of $\text{Exp}(X^2)$.

**Theorem 3.1.** Let $X$ be a compact space, $G \subset H(X)$. Let $R \subset \text{Exp}(X^2)$ be the compact $G$-space defined above. The family consisting of the single compact $G$-space $\text{Exp}R$ is representative.

In other words, $S(G)$ is isomorphic to the enveloping semigroup of $\text{Exp}R$.

**Proof.** Let $A, B \subset G$ be far from each other (that is, $1_G$ is not in the closure of $BA^{-1}$). In virtue of proposition 2.4, it suffices to find $p \in Y = \text{Exp}R$ such that $Ap$ and $Bp$ have disjoint closures in $Y$.

Let $p$ be the closure of the set $\{\Gamma_g : g \in A^{-1}\}$ in the space $\text{Exp}(X^2)$. Then $p$ is a closed subset of $R$ and hence $p \in Y$. We claim that $p$ has the required property: $Ap$ and $Bp$ have disjoint closures in $Y$ or, which is the same, in $\text{Exp}\text{Exp}(X^2)$.

There exist a continuous pseudometric $d$ on $X$ and $\delta > 0$ such that

$$\forall f \in A \forall g \in B \exists x \in X \ (d(gf^{-1}(x), x) \geq \delta).$$

Let $\Delta \subset X^2$ be the diagonal. Let $C \subset X^2$ be the closed set defined by

$$C = \{(x, y) \in X^2 : d(x, y) \geq \delta\}.$$  

Let $K \subset \text{Exp}X^2$ be the closed set defined by

$$K = \{F \subset X^2 : F \text{ meets } C\}.$$  

Consider the closed sets $L_1, L_2 \subset \text{Exp}\text{Exp}(X^2)$ defined by

$$L_1 = \{q \subset \text{Exp}(X^2) : q \text{ is closed and } \Delta \in q\}$$

and

$$L_2 = \{q \subset \text{Exp}(X^2) : q \text{ is closed and } q \subset K\}.$$  

Since $\Delta \notin K$, the sets $L_1$ and $L_2$ are disjoint. It suffices to verify that $Ap \subset L_1$ and $Bp \subset L_2$. 

The first inclusion is immediate: if $g \in A$, then for $h = g^{-1}$ we have $\Delta = g\Gamma_h \in gp$, hence $gp \in L_1$. Thus $Ap \subset L_1$. We now prove that $Bp \subset L_2$. Let $g \in B$. If $f \in A$ and $h = f^{-1}$, there exists $x \in X$ such that $d(gh(x), x) \geq \delta$, which means that $\Gamma_{gh}$ meets $C$. Hence $g\Gamma_h = \Gamma_{gh} \in K$. It follows that the closed set $g^{-1}K$ contains the set $\{\Gamma_h : h \in A^{-1}\}$ and hence also its closure $p$. In other words, $gp \subset K$ and hence $gp \in L_2$. □

As noted in Section 1, Theorem 1.1 follows from Theorems 2.2 and 3.1.

Combining Theorems 2.3 and 3.1, we obtain the following generalization of Theorem 1.1:

**Theorem 3.2.** Let $X$ be a compact space, $G$ a subgroup of $H(X)$. Let $R$ be the same as in Theorems 1.1 and 3.1. Let $\mathcal{F}$ be the family of all minimal closed $G$-subspaces of $\text{Exp} R$. Then $\mathcal{M}(G)$ is isomorphic to a subspace of a product of members of $\mathcal{F}$ (some factors may be repeated).

4. **Proof of Theorem 1.5**

Theorem 3.1 implies that for any subgroup $G \subset H(X)$ the one-point family $\{\text{ExpExp}(X^2)\}$ is representative (recall that we consider the trivial action on the first factor $X$). I do not know whether $X^2$ can be replaced here by $X$. On the other hand, the following holds:

**Theorem 4.1.** Let $X$ be a compact space, $G$ a subgroup of $H(X)$. The sequence $\{\text{Exp}(\text{Exp}X^n)\}_{n=1}^\infty$ of compact $G$-spaces is representative.

This is a special case of a more general theorem:

**Theorem 4.2.** Let $X$ be a compact space, $G$ a subgroup of $H(X)$. Let $Y_n$ be a closed $G$-invariant subset of $(\text{Exp}X)^n$ $(n = 1, 2, \ldots)$ such that $\bigcup_{n=1}^\infty Y_n$ contains arbitrarily fine covers of $X$. Then the sequence $\{\text{Exp}Y_n\}_{n=1}^\infty$ of compact $G$-spaces is representative.

**Proof.** Let $A, B \subset G$ be two sets that are far from each other with respect to the right uniformity. In virtue of proposition 2.4, it suffices to find $n$ and a point $p \in \text{Exp}Y_n$ such that $Ap$ and $Bp$ have disjoint closures in $\text{Exp}Y_n$ or, which is the same, in $Z_n = \text{Exp}((\text{Exp}X)^n)$. 
There exist a continuous pseudometric $d$ on $X$ and $\delta > 0$ such that $A$ and $B$ are $(d, 2\delta)$-far from each other, in the sense that
\[
\forall f \in A \forall g \in B \exists x \in X \ (d(f(x), g(x)) > 2\delta).
\]
The assumption that $\bigcup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers implies that we can find $n \geq 1$ and closed sets $C_1, \ldots, C_n \subset X$ of $d$-diameter $\leq \delta$ such that $(C_1, \ldots, C_n) \in Y_n$ and $\bigcup_{i=1}^{n} C_i = X$. For each $g \in G$, let $F_g = (g^{-1}(C_1), \ldots, g^{-1}(C_n)) \in (\text{Exp} X)^n$. Since $Y_n$ is $G$-invariant, we have $F_g \in Y_n$. Let $p$ be the closure of the set $\{F_g : g \in A\}$ in the space $(\text{Exp} X)^n$. Then $p \in \text{Exp} Y_n$. We claim that $p$ has the required property: $Ap$ and $Bp$ have disjoint closures in $Z_n$.

Let $D_i = \{x \in X : d(x, C_i) \geq \delta\}$, $i = 1, \ldots, n$. Consider the closed sets $K_1, K_2 \subset (\text{Exp} X)^n$ defined by
\[
K_1 = \{(F_1, \ldots, F_n) \in (\text{Exp} X)^n : F_i \subset C_i, \ i = 1, \ldots, n\}
\]
and
\[
K_2 = \{(F_1, \ldots, F_n) \in (\text{Exp} X)^n : F_i \text{ meets } D_i \text{ for some } i = 1, \ldots, n\}.
\]
Consider the closed sets $L_1, L_2 \subset Z_n$ defined by
\[
L_1 = \{q \subset (\text{Exp} X)^n : q \text{ is closed and } q \text{ meets } K_1\}
\]
and
\[
L_2 = \{q \subset (\text{Exp} X)^n : q \text{ is closed and } q \subset K_2\}.
\]
Clearly $K_1$ and $K_2$ are disjoint, hence $L_1$ and $L_2$ are disjoint as well. It suffices to verify that $Ap \subset L_1$ and $Bp \subset L_2$.

The first inclusion is immediate: if $g \in A$, then $F_g \in p$ and $gF_g = (C_1, \ldots, C_n) \in K_1 \cap gp$, hence $gp$ meets $K_1$ and $gp \in L_1$.

We now prove that $Bp \subset L_2$. Let $h \in B$. If $g \in A$, we can find $x \in X$ such that $d(g(x), h(x)) > 2\delta$ and an index $i$, $1 \leq i \leq n$, such that $g(x) \in C_i$. Since diam $C_i \leq \delta$, we have $h(x) \in D_i$ and therefore $h(x) \in hg^{-1}(C_i) \cap D_i \neq \emptyset$. It follows that $hF_g = (hg^{-1}(C_1), \ldots, hg^{-1}(C_n)) \in K_2$. This holds for every $g \in A$, and thus we have shown that the closed set $h^{-1}K_2 \subset (\text{Exp} X)^n$ contains the set $\{F_g : g \in A\}$ and hence also its closure $p$. In other words, $hp \subset K_2$ and hence $hp \in L_2$.

\[\square\]

Theorem 1.5 follows from Theorems 4.2 and 2.2.

Combining Theorems 2.3 and 4.2, we obtain the following generalization of Theorem 1.5:
Theorem 4.3. Let $X$ be a compact space, $G$ a subgroup of $H(X)$. Let $Y_n$ be a closed $G$-invariant subset of $(\text{Exp} \ X)^n$ ($n = 1, 2, \ldots$) such that $\bigcup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of $X$. Let $\mathcal{F}$ be the family of all (up to an isomorphism) minimal closed $G$-subspaces of $\text{Exp} Y_n$, $n = 1, 2, \ldots$. Then $\mathcal{M}(G)$ is isomorphic to a subspace of a product of members of $\mathcal{F}$ (some factors may be repeated).

REFERENCES


321 Morton Hall, Department of Mathematics, Ohio University, Athens, Ohio 45701, USA
E-mail address: uspensk@math.ohiou.edu