ON TWO PROBLEMS CONCERNING
TOPOLOGICAL CENTERS

by

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ABSTRACT. Let $\Gamma$ be an infinite discrete group and $\beta \Gamma$ its Čech-Stone compactification. Using the well known fact that a free ultrafilter on an infinite set is nonmeasurable, we show that for each element $p$ of the remainder $\beta \Gamma \setminus \Gamma$, left multiplication $L_p : \beta \Gamma \to \beta \Gamma$ is not Borel measurable. Next assume that $\Gamma$ is abelian. Let $D \subset \ell^\infty(\Gamma)$ denote the subalgebra of distal functions on $\Gamma$ and let $D = \Gamma^D = |D|$ denote the corresponding universal distal (right topological group) compactification of $\Gamma$. Our second result is that the topological center of $D$ (i.e. the set of $p \in D$ for which $L_p : D \to D$ is a continuous map) is the same as the algebraic center and that for $\Gamma = \mathbb{Z}$, this center coincides with the canonical image of $\Gamma$ in $D$.

1. Introduction

This short note is a direct outcome of the topology conference held in Castellón in the summer of 2007. I was presented during the conference with two problems relating to the topological center of certain right topological semigroups. (A compact semigroup $A$ such that for each $p \in A$ the corresponding right multiplication $R_p : q \mapsto qp$ is continuous is called a right topological semigroup. The collection of elements $p \in A$ for which the corresponding left multiplication $L_p : q \mapsto pq$ is continuous is called the topological center of $A$.) The first was a question of Michael Megrelishvili:

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Given an infinite discrete group $\Gamma$, which are the elements of $\beta\Gamma$ for which $L_p : \beta\Gamma \to \beta\Gamma$ is a Baire class 1 map? (It is known that the topological center of $\beta\Gamma$ is exactly $\Gamma$ itself, considered as a subset of $\beta\Gamma$, see e.g. [5].) The second problem is due to Mahmoud Filali: If $D = D(\Gamma)$ is the universal distal Ellis group of $\Gamma$, identify the topological center of $D$.

I present here a complete answer to Megrelishvili’s problem, based on the well known result that a free ultrafilter on an infinite set is nonmeasurable, and an answer to Filali’s problem in the case $\Gamma = \mathbb{Z}$, the group of integers.

The interested reader is referred to [3, chapter 1] and the bibliography list thereof, for more information on the abstract theory of topological dynamics, and to [5] for information concerning $\beta\Gamma$.

I thank both Megrelishvili and Filali for addressing to me these nice problems. I also thank the organizers of the Castellón meeting for the formidable effort they put into the details of the conference and for their warm hospitality.

2. On the center of $\beta\Gamma$

We consider the Čech-Stone compactification $\beta\Gamma$ as the collection of ultrafilters on $\Gamma$. The topology on $\beta\Gamma$ is given by the collection of basic clopen sets $\{V_A : A \subset \Gamma\}$, where $V_A = \{p \in \beta\Gamma, A \in p\}$. We also recall that the semigroup product on $\beta\Gamma$ is defined by

$$A \in pq \iff \{\gamma \in \Gamma : \gamma^{-1}A \in q\} \in p.$$ 

For more details see e.g. [2].

**Theorem 2.1.** Let $\Gamma$ be an infinite countable discrete group and $\beta\Gamma$ its Čech-Stone compactification. For each element $p$ of the remainder $\beta\Gamma \setminus \Gamma$, left multiplication $L_p : \beta\Gamma \to \beta\Gamma$ is not Borel measurable.

**Proof.** Let $\mathcal{P}(\Gamma)$ denote the collection of all subsets of $\Gamma$. Let $\Omega = \{0, 1\}^\Gamma$ and let $\chi : \mathcal{P}(\Gamma) \to \Omega$ denote the canonical map $\chi(A) = 1_A$ for $A \subset \Gamma$. We regard $\Omega$ as a compact space and let $\mathcal{B}$ denote its Borel $\sigma$-algebra. Let $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^\Gamma$ denote the product probability measure on $\Omega$ and let $\mathcal{B}_\mu$ denote the completion of $\mathcal{B}$ with respect to $\mu$. 

A well known and easy fact, which for completeness we will reproduce below (Lemma 2.3), is that a free ultrafilter on an infinite set is nonmeasurable: Viewing an element \( p \in \beta \Gamma \setminus \Gamma \) as an ultrafilter on \( \Gamma \), the collection \( \{ \chi(A) : A \in p \} \subset \Omega \), is not \( \mu \)-measurable; i.e. not an element of \( B_\mu \). In particular it is not a Borel subset of \( \Omega \). (In fact, it is not even Baire measurable, [6] and [7].)

The compact space \( \Omega \) becomes a dynamical system when we let \( \Gamma \) act on it by permuting the coordinates:

\[
\gamma \omega(\gamma') = \omega(\gamma^{-1} \gamma').
\]

Of course the measure \( \mu \) is \( \Gamma \)-invariant, but we will not need this fact. The action of \( \Gamma \) on \( \Omega \) extends to an action of \( \beta \Gamma \) in the natural way and we write \( p \omega \) for the image of \( \omega \in \Omega \) under \( p \in \beta \Gamma \). (In fact via this “action” \( \beta \Gamma \) is identified with the enveloping semigroup of the system \((\Omega, \Gamma)\), see [3, chapter 1].)

For \( A \subset \Gamma \) and \( p \in \beta \Gamma \) set

\[
p \star A = \{ \gamma \in \Gamma : \gamma A^{-1} \in p \}
\]

and check that \( \gamma 1_A = 1_{\gamma A} \) and \( p \chi(A) = p 1_A = 1_{p \star A} = \chi(p \star A) \).

Moreover if \( q \in \beta \Gamma \) then

\[
pq \star A = p \star (q \star A).
\]

For convenience I sometimes identify a subset \( A \subset \Gamma \) with the corresponding element \( 1_A = \chi(A) \) in \( \Omega \).

Let \( \pi_e : \Omega \to \{0, 1\} \) denote the projection on the \( e \)-component of \( \Omega \). Here \( e \) is the neutral element of \( \Gamma \). Let \( \omega_0 = 1_D \) be some fixed element of \( \Omega \) whose \( \Gamma \) orbit is dense in \( \Omega \). Let \( \psi : \gamma \mapsto \gamma \omega_0 \) be the orbit map and let \( \hat{\psi} \) denote its unique extension to \( \beta \Gamma \). Thus \( \hat{\psi}(q) = q \omega_0 \) for \( q \in \beta \Gamma \).

Consider the map \( L_p : \beta \Gamma \to \beta \Gamma \) and write \( \phi : \beta \Gamma \to \{0, 1\} \) for the map \( \phi = \pi_e \circ \hat{\psi} \circ L_p \). (Thus \( \phi(q) = (pq \omega_0)(e) \).) We define \( J : \Omega \to \Omega \) by \( (J(\omega))(\gamma) = \omega(\gamma^{-1}) \).

We have

\[
Q := \phi^{-1}(1) = \{ q \in \beta \Gamma : pq \omega_0(e) = 1 \}.
\]
Now $pq_0(e) = 1$ iff $e \in \chi^{-1}(pq_0)$ hence
\[ Q = \{ q \in \beta \Gamma : pq_0(e) = 1 \} \]
\[ = \{ q \in \beta \Gamma : e \in p \ast q_0 \} \]
\[ = \{ q \in \beta \Gamma : e \in p \ast q \ast D \} \]
\[ = \{ q \in \beta \Gamma : e \in \{ \gamma \in \Gamma : \gamma(q \ast D)^{-1} \in p \} \} \]
\[ = \{ q \in \beta \Gamma : (q \ast D)^{-1} \in p \} \]
\[ = \{ q \in \beta \Gamma : J(q_0) \in p \}. \]

Thus $(\hat{J} \circ \hat{\psi})(Q) = P$ and since also $(\hat{\psi}^{-1} \circ J^{-1})(p) = Q$ we conclude that $Q = \phi^{-1}(1)$ is not Borel measurable in $\beta \Gamma$. Finally, since also $Q = L_p^{-1}(\{ q \in \beta \Gamma : (q_0)(e) = 1 \})$ we see that $L_p$ is not Borel measurable. \hfill \square

In the next two lemmas we follow [7] (see also [1]). Let $I$ be an infinite set, $\Omega = \{ 0, 1 \}^I$. As above, we identify subsets $A$ of $I$ with their characteristic functions $1_A \in \Omega = \{ 0, 1 \}^I$ and, accordingly, filters on $I$ with subsets of $\Omega$. Let $\phi : \Omega \rightarrow \Omega$ denote the “flip” function defined by $\phi(\omega)_i = 1 - \omega_i$, $i \in I$. We consider the measure space $(\Omega, \Sigma_\lambda, \lambda)$, where $\Omega = \{ 0, 1 \}^I$, $\lambda$ is the Bernoulli measure $\lambda = (\frac{1}{2}(\delta_0 + \delta_1))^I$, and $\Sigma_\lambda$ denotes the completion of the Borel $\sigma$-algebra with respect to $\lambda$. As usual we use the notation $\lambda_*$ and $\lambda^*$ for the induced inner and outer measures.

The assertions of the next lemma are easily verified.

**Lemma 2.2.**
1. The involution $\phi$ is measurable and it preserves $\lambda$.
2. For $A \subset I$ we have $\phi(1_A) = 1_{A^c}$.
3. If $\mathcal{F}$ is a filter on $I$ then $\phi \mathcal{F} \cap \mathcal{F} = \emptyset$.
4. If $\mathcal{F}$ is a free filter on $I$ (i.e. $\bigcap \mathcal{F} = \emptyset$) then, considered as a collection of subsets of $\{ 0, 1 \}^I$ it is a “tail event”, that is, for every finite subset $F \subset I$, $\mathcal{F} = \{ 0, 1 \}^F \times \mathcal{F}'$, with $F' \subset \{ 0, 1 \}^{I \setminus F}$.
5. A filter $\mathcal{F}$ on $I$ is an ultrafilter iff $\phi(\mathcal{F}) \cup \mathcal{F} = \Omega$.

**Lemma 2.3.** Let $\mathcal{F}$ be a free filter on $I$. Then
1. $\lambda_*(\mathcal{F}) = 0$.
2. $\lambda^*(\mathcal{F}) \in \{ 0, 1 \}$.
3. $\lambda^*(\mathcal{F}) = 1$ if $\mathcal{F}$ is an ultrafilter.
4. A free filter $\mathcal{F}$ is measurable iff $\lambda^*(\mathcal{F}) = 0$, and nonmeasurable iff $\lambda^*(\mathcal{F}) = 1$. In particular, every free ultrafilter is nonmeasurable.

Proof. If $\mathcal{F}$ is a free filter on $I$ and $\mathcal{G} \subset \mathcal{F}$ is a measurable tail event then it has measure either 0 or 1. Thus $\lambda^*(\mathcal{F}) \in \{0, 1\}$. This proves part 2. We also have $\lambda_*(\mathcal{F}) \in \{0, 1\}$ and since $\phi(\mathcal{F}) \cap \mathcal{F} = \emptyset$ it follows that

$$1 = \lambda(\Omega) \geq \lambda_*(\phi\mathcal{F}) + \lambda_*(\mathcal{F}) = 2\lambda_*(\mathcal{F}).$$

We conclude that $\lambda_*(\mathcal{F}) = 0$, proving part 1. If $\mathcal{F}$ is an ultrafilter then $\mathcal{F} \cup \phi\mathcal{F} = \{0, 1\}^I$ and we conclude that

$$1 = \lambda(\Omega) \leq \lambda^*(\phi\mathcal{F}) + \lambda^*(\mathcal{F}) = 2\lambda^*(\mathcal{F}),$$

whence $\lambda^*(\mathcal{F}) = 1$. This proves part 3. Part 4 is now clear. □

3. ON THE CENTER OF $\Gamma^D$, THE UNIVERSAL DISTAL ELLIS GROUP OF $\Gamma$

Let $\Gamma$ be a discrete abelian group. Let $\mathcal{D}$ denote the closed $\Gamma$-invariant subalgebra of (complex valued) distal functions in $\ell^\infty(\Gamma)$. Let $D = \Gamma^D = |\mathcal{D}|$ denote the corresponding Gelfand space. It is well known that $D$ is the largest right topological group compactification of $\Gamma$.

**Theorem 3.1.** Let $\Gamma$ be an infinite discrete abelian group. The topological center of $D = \Gamma^D$ is the same as the algebraic center and, when $\Gamma = \mathbb{Z}$, it also coincides with the canonical image of $\Gamma$ in $D$.

**Proof.** In order to simplify our notation we will identify elements of $\Gamma$ with their images in $D$. The coincidence of the topological and algebraic centers of $D$ is easy: Suppose first that $p \in D$ is in the algebraic center of this group. Then, as right multiplication is always continuous, we have for any convergent net $q_\alpha \to q$ in $D$

$$pq = qp = \lim q_\alpha p = \lim p q_\alpha,$$

i.e. $L_p : D \to D$ is continuous.
Conversely, assume that \( p \) is in the topological center; i.e. \( L_p : D \to D \) is continuous. We note that if \( q \) is an element of \( D \) then \( \gamma q = q\gamma \) for every \( \gamma \in G \). In fact choosing a convergent net \( \Gamma \ni \gamma_\alpha \to q \), by the commutativity of \( \Gamma \),

\[
\gamma q = \gamma \lim \gamma_\alpha = \lim \gamma \gamma_\alpha = \lim \gamma_\alpha \gamma = q\gamma.
\]

Now, with this in mind, we have

\[
pq = p \lim \gamma_\alpha = \lim p \gamma_\alpha = \lim \gamma_\alpha p = qp,
\]

so that \( p \) is indeed an element of the center.

Now to the more delicate task of showing that this center coincides with \( \Gamma \). Let \( p \in D \) be a central element. If \( p \notin \Gamma \) then there exists a metric minimal distal dynamical system \((Y, \Gamma)\) and a point \( y_0 \in Y \) such that

\[
(3.1) \quad py_0 \notin \Gamma y_0.
\]

By assumption the map \( L_p : D \to D \) is continuous (in fact a homeomorphism) and as we have seen it also commutes with every element of \( \Gamma \). In other words, \( L_p \) is an automorphism of the system \((D, \Gamma)\). Now the dynamical system \((D, \Gamma)\) is the universal distal system and therefore, it admits a unique homomorphism of dynamical systems \( \hat{\phi} : (D, \Gamma) \to (E(Y, \Gamma), \Gamma) \) of \( E = E(Y, \Gamma) \) (which by a theorem of Ellis is in fact a group) such that \( \hat{\phi}(e_D) = e_E \). Now the map \( \phi : y \mapsto \hat{\phi}(p)y \) (which we write simply as \( y \mapsto py_0 \)) is a homomorphism \( \phi : (D, \Gamma) \to (Y, \Gamma) \) with \( \phi(e) = y_0 \). If \( y_\alpha \to y \) is a convergent net in \( Y \) then there are \( q_\alpha \in D \) with \( y_\alpha = q_\alpha y_0 \). With no loss of generality we have \( q_\alpha \to q \) in \( D \), so that in particular \( y = \lim y_\alpha = \lim q_\alpha y_0 = qy_0 \). Now we see that

\[
py = pqy_0 = (p \lim q_\alpha)y_0
= (\lim p q_\alpha)y_0 = \lim p(q_\alpha y_0)
= \lim p y_\alpha.
\]

Thus \( p \) acts continuously on \( Y \). Since also \( p\gamma = \gamma p \) for every \( \gamma \in \Gamma \) we conclude that \( p \) is an automorphism of the system \( (Y, \Gamma) \).

Note that this argument shows that \( p \text{ acts as an automorphism of every factor of } (D, \Gamma) \). Therefore, our proof will be complete when we find a minimal distal dynamical system \((X, \Gamma)\) extending \((Y, \Gamma)\), say \( \pi : (X, \Gamma) \to (Y, \Gamma) \), where \( p \) is not an automorphism.
At this stage, in order to be able to use a method of construction developed by Glasner and Weiss in [4], we specialize to the case \( \Gamma = \mathbb{Z} \). In particular the system \((Y, \Gamma)\) which was singled out in the above discussion has now the form \((Y, T)\) where \( T : Y \to Y \) is a self homeomorphism of \( Y \) determined by the element \( 1 \in \mathbb{Z} \). Of course we can assume that \( Y \) is non-periodic (i.e. infinite).

The following construction is a special case of a general setup designed in [4] for providing minimal extensions of a given non-periodic minimal \( \mathbb{Z} \)-system \((Y, T)\). We refer the reader to [4] for more details.

Set \( X = Y \times K \) where \( K \) denotes the circle group \( K = S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( \Theta \) be the family of continuous maps \( \theta : Y \to K \).

For each \( \theta \in \Theta \) let \( G_\theta : X \to X \) be the map \( G_\theta(y, z) = (y, z\theta(y)) \) and \( S_\theta = G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \). Thus

\[
S_\theta : X \to X, \quad S_\theta(y, z) = (Ty, z\theta(\phi(Ty)^{-1})).
\]

Form the collection \( S(T) = \{ G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta : \theta \in \Theta \} \).

Theorem 1 of [4] ensures that in the set \( S(T) \) (closure with respect to the uniform convergence topology in \( \text{Homeo}(X) \)) there is a dense \( G_\delta \) subset \( \mathcal{R} \) such that for every \( R \in \mathcal{R} \) the system \((X, R)\) is minimal, distal, and the projection map \( \pi : X \to Y \) is a homomorphism of dynamical systems \((\pi R(y, z) = T\pi(y, z) = Ty)\). Note that every \( R \in S(T) \) has the form

\[
R = T_\phi : X \to X, \quad \text{where} \quad T_\phi(y, z) = (Ty, z\phi(y)),
\]

for some continuous map \( \phi : Y \to K \). We will often use the fact that for \( n \in \mathbb{N} \) the \( n \)-th iteration of \( T_\phi \) has the form

\[
T_n \phi(y, z) = (T^n y, z\phi_n(y)), \quad \text{where} \quad \phi_n(y) = \phi(T^{n-1}y) \cdots \phi(Ty)\phi(y).
\]

Note that when \( \phi \) has the very special form \( \phi(y) = \theta(\phi(Ty)^{-1}\theta(y)) \) for some continuous \( \theta : Y \to K \), the equation (3.3) collapses:

\[
\phi_n(y) = \theta(T^n y)^{-1}\theta(y), \quad \text{hence} \quad S_n \theta(y, z) = (T^n y, z\theta(T^n y)^{-1}\theta(y)).
\]

We temporarily fix an element \( R = T_\phi \in \mathcal{R} \). As observed above, the element \( p \in D \) defines an automorphism of the system \((X, T_\phi)\); moreover we have for every \( x = (y, z) \in X \):

\[
\pi(px) = p\pi(x) = py.
\]
This last observation implies that \( p : X \to X \) has the form \( p(y, z) = (py, \omega(y, z)) \) for some continuous map \( \omega : Y \times K \to K \).

**Lemma 3.2.** The function \( \omega \) has the form \( \omega(y, z) = z\psi(y) \) for some continuous map \( \psi : Y \to K \), whence
\[
p(y, z) = (py, \psi(y)).
\]

**Proof.** There exists a net \( \{n_\nu\}_{\nu \in I} \) in \( Z \) such that \( p = \lim n_\nu \) in \( D \).

Thus, for every \((y, z) \in X\)
\[
p(y, z) = \lim_n T^n_n y = \lim_n T^n_n z = \lim_n \phi_n(y) \psi(Ty)
\]
where the point-wise limit \( \psi(y) := \lim \phi_n(y) \) is necessarily a continuous function. \( \square \)

The commutation relation \( pT_\phi = T_\phi p \) now reads:
\[
pT_\phi(y, z) = p(Ty, z\psi(y)) = (py, z\psi(Ty))
\]
\[
= T_\phi(p, z) = T_\phi(py, z\psi(y))
\]
\[
= (Tpy, z\psi(y)\psi(py)).
\]

In turn this implies:
\[
\phi(y)\psi(Ty) = \psi(y)\phi(py).
\]

Similarly the commutation relations \( pT^n_\phi = T^n_\phi p \) yield:
\[
\phi_n(y)\psi(T^n y) = \psi(y)\phi_n(py).
\]

Next consider any sequence \( n_i \not\to \infty \) such that
- \( \lim T^{n_i} y_0 = y_0 \),
- \( \lim \phi_{n_i}(y_0) = z' \), and
- \( \lim \phi_{n_i}(py_0) = z'' \).

Applying (3.6) and taking the limit as \( i \to \infty \) we get \( \psi(y_0)z'' = z'\psi(y_0) \), whence necessarily also \( z' = z'' \).

The proof of Theorem 3.1 will be complete when we next show that for a residual subset \( R_1 \) of \( \overline{S(T)} \), we have \( \lim \phi_{n_i}(y_0) = z' \neq z'' = \lim \phi_{n_i}(py_0) \), whenever \( R = T_\phi \in R_1 \). Then for any element \( T_\phi \in R \cap R_1 \), \( (X, T_\phi) \) will serve as a minimal distal system where \( p \) is not an automorphism.

**Proposition 3.3.** For a given sequence \( n_i \not\to \infty \) with \( \lim T^{n_i} y_0 = y_0 \), the set
\[
R_1 = \{ T_\phi \in \overline{S(T)} : \forall i \exists j > i, \ |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 \}
\]
is a residual subset of \( \overline{S(T)} \).
Proof. For \( i \in \mathbb{N} \) and \( \eta > 0 \) set
\[
E_{i,\eta} = \{ T_\theta \in \overline{S(T)} : \exists j > i, \; |\phi_{n_j}(y_0) - \phi_{n_j}(y_0)| > 1 + \eta \}.
\]
Clearly \( E_{i,\eta} \) is an open subset of \( \overline{S(T)} \) and for \( i < k \) we have \( E_{k,\eta} \subset E_{i,\eta} \).

Lemma 3.4. Given \( i \) and \( \eta > 0 \), for every \( \theta_0 \in \Theta \) there exists an \( i_0 > i \) such that
\[
G_{\theta_0}^{-1}E_{i_0,\eta}G_{\theta_0} \subset E_{i_0,\eta/2}.
\]

Proof. Fix \( \theta_0 \in \Theta \). For sufficiently large \( i_0 \), for all \( j > i_0 \) the distances \( d(T^{n_j}y_0, y_0) \) and \( d(T^{n_j}y_0, y_0) \) are so small that
\[
|\theta(T^{n_j}y_0) - \phi_{n_j}(y_0)| > 1 + \eta/2
\]
holds whenever
\[
|\phi_{n_j}(y_0) - \phi_{n_j}(y_0)| > 1 + \eta.
\]

We will show that \( E_{i,\eta} \) is also dense in \( \overline{S(T)} \). For this it suffices to show that \( G_{\theta_0}^{-1} \circ (T \times \text{id}) \circ G_{\theta_0} \in E_{i,\eta} \) for every \( \theta \in \Theta \), i.e. \( T \times \text{id} \in G_{\theta_0}E_{i,\eta}G_{\theta_0}^{-1} \).

Now for a fixed \( \theta_0 \) there is by Lemma 3.4, an \( i_0 > i \) with \( G_{\theta_0}^{-1}E_{i_0,2\eta}G_{\theta_0} \subset E_{i_0,\eta} \), hence it suffices to show that \( T \times \text{id} \in \overline{E_{i_0,2\eta}} \), since then
\[
T \times \text{id} \subset \overline{E_{i_0,2\eta}} \subset \overline{G_{\theta_0}E_{i_0,\eta}G_{\theta_0}^{-1}} \subset \overline{G_{\theta_0}E_{i_0,\eta}G_{\theta_0}^{-1}}.
\]
Finally the next lemma will prove this last assertion and therefore also the density of \( E_{i,\eta} \) for every \( i \) and \( 0 < \eta < 1 \).

Lemma 3.5. Given \( i \in \mathbb{N}, 0 < \eta < 1 \) and \( \varepsilon > 0 \) there exists \( \theta \in \Theta \) such that
1. \( d(T \times \text{id}, G_{\theta}^{-1} \circ (T \times \text{id}) \circ G_{\theta}) < \varepsilon \).
2. \( G_{\theta}^{-1} \circ (T \times \text{id}) \circ G_{\theta} \in E_{i,\eta} \).

Proof. Let \( I = [0, 1] \) and set \( h(0) = h(1/3) = h(2/3) = 1, \; h(1) = -1 \) and extend this function in an arbitrary way to a continuous \( h : I \to S^1 \). Choose \( \delta > 0 \) such that \( |t-s| < \delta \) implies \( |h(t) - h(s)| < \varepsilon \). Let \( m \in \mathbb{N} \) be such that \( 2/m < \delta \). Let \( U_1 \) and \( U_2 \) be open neighborhoods of \( y_0 \) and \( py_0 \), respectively, in \( Y \) such that for \( s = 1, 2 \), the sets \( U_s, TU_s, \ldots, T^{m-1}U_s \) are mutually disjoint. (Here we use the facts that \( Y \) is infinite and that \( py_0 \notin \{ T^jy_0 : j \in \mathbb{Z} \} \) (3.1).) Choose \( k > i \) so that \( T^{nk}y_0 \in U_1 \) and \( T^{nk}py_0 \in U_2 \).
Let $K_s \subset U_s$, $s = 1, 2$, be Cantor sets such that $y_0, T^{n_k}y_0 \in K_1$ and $p y_0, T^{n_k}p y_0 \in K_2$.

Next define:

$$g(y_0) = 0, \quad g(T^{n_k}y_0) = 1/3, \quad g(p y_0) = 2/3, \quad g(T^{n_k}p y_0) = 1$$

and extend this function in an arbitrary way to a continuous function $g : K_1 \cup K_2 \to S^1$. We now extend $g$ to the set $\bigcup_{j=0}^{m-1} T^j (K_1 \cup K_2)$ by setting $g(y) = g(T^{-j}y)$ for $y \in T^j(K_1 \cup K_2)$. Extend $g$ continuously over all of $Y$ in an arbitrary way.

Set

$$\tilde{g}(y) = \frac{1}{m} \sum_{j=0}^{m-1} g(T^jy).$$

Clearly $\tilde{g} | (K_1 \cup K_2) = g | (K_1 \cup K_2)$, so that

$$\tilde{g}(y_0) = 0, \quad \tilde{g}(T^{n_k}y_0) = 1/3, \quad \tilde{g}(p y_0) = 2/3, \quad \tilde{g}(T^{n_k}p y_0) = 1.$$ 

Finally define $\theta : Y \to S^1$ by $\theta(y) = h(\tilde{g}(y))$. Note that

$$(3.7) \quad \theta(y_0) = \theta(T^{n_k}y_0) = \theta(p y_0) = 1, \quad \text{and} \quad \theta(T^{n_k}p y_0) = -1.$$ 

Now

$$G^{-1}_{\theta} \circ (T \times \text{id}) \circ G_{\theta}(y, z) = (T y, z \theta(T y)^{-1}\theta(y)) = (T y, z h(\tilde{g}(T y))^{-1}h(\tilde{g}(y))).$$

But

$$|\tilde{g}(T y) - \tilde{g}(y)| < 2/m < \delta,$$

hence $|h(\tilde{g}(T y))^{-1}h(\tilde{g}(y)) - 1| < \varepsilon$ and therefore also

$$d(T \times \text{id}, G^{-1}_{\theta} \circ (T \times \text{id}) \circ G_{\theta}) < \varepsilon.$$ 

This proves part (1) of the lemma and we now proceed to prove part (2). We have to show that $G^{-1}_{\theta} \circ (T \times \text{id}) \circ G_{\theta} \in E_i$. But this map has the form

$$S_{\theta} : X \to X, \quad S_{\theta}(y, z) = G^{-1}_{\theta} \circ (T \times \text{id}) \circ G_{\theta} = (T y, z \theta(T y)^{-1}\theta(y)),$$

so that, by (3.4), we have to show that there exists $j > i$ with

$$|\theta(T^{n_j}y_0)^{-1}\theta(y_0) - \theta(T^{n_j}p y_0)^{-1}\theta(p y_0)| > 1 + \eta.$$ 

Since, by the choice of $\theta$ (3.7), we have

$$|\theta(T^{n_k}y_0)^{-1}\theta(y_0) - \theta(T^{n_k}p y_0)^{-1}\theta(p y_0)| = |1 - (-1)| = 2 > 1 + \eta,$$

this completes the proof of the lemma. □
To conclude the proof of Proposition 3.3 observe that, for instance, the dense $G_δ$ set $\bigcap_{i=1}^{\infty} E_{i,1/2}$ is contained in $\mathcal{R}_1$. □

This also concludes the proof of Theorem 3.1. □

REFERENCES


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