A Note on Ditopological Texture Spaces

by

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Abstract. In this study, we present the hereditary separation properties of plain ditopological texture spaces for induced subtextures. Brown and his team proved that the complete biregularity is productive. Here, using the concept of induced subtexture, we show that the converse of this result is true for plain texture spaces, namely if a product of plain ditopological texture spaces is completely biregular, then all factor spaces are also completely biregular.

1. Introduction

From the motivational point of view, textures were first considered only as a point–base setting for fuzzy sets [see e.g. 9], but in recent papers [5,12] it is observed that they are in fact $C$–spaces [17], and equivalently $T_0$ core spaces [16] or $T_0$ topological spaces with injective hulls [4] as objects of the category dfTex where the morphisms are difunctions. Further, the highly economic structures of textures can be used in obtaining well–known constructions as Wallman or Alexandroff compactifications and this may play an important role to determine the structures of principal examples in main stream topology [2,15]. On the other hand, in a textural discussion, the symmetry property of uniform spaces corresponds to a kind of complementation of direlations [19]. The basic separation properties of ditopological textures are studied extensively in [11] and it is proved that the complete biregularity is productive.

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In this study, we present the hereditary separation properties of ditopological texture spaces and using induced subtexture, we show that if a product of plain ditopological texture spaces is completely biregular, then all factor spaces are also completely biregular in the category of dfPDitop. Now let us recall some basic concepts on textures from [9] and [10]. A texturing on a set $S$ is a point separating, complete, completely distributive lattice $S$ of subsets of $S$ with respect to inclusion which contains $S, \emptyset$ and, for which arbitrary meet coincides with intersection and finite joins coincide with union. Then $(S, S)$ is called a texture space. The sets

$$P_s = \bigcap \{A \mid s \in A \in S\} \mbox{ and } Q_s = \bigvee \{P_t \mid s \notin P_t\}$$

are known as $p$–sets and $q$–sets and they are important tools for textures as we will see in the sequel. If for all $s \in S$, we have $P_s \subsetneq Q_s$, then $(S, S)$ is called a \textit{plain texture space}. This is equivalent to say that $(S, S)$ is closed under arbitrary unions. Now let $(S, S)$, $(T, T)$ be texture spaces. Consider the product texture $\mathcal{P}(S) \otimes T$ of the texture spaces $(S, \mathcal{P}(S))$ and $(T, T)$ and denote the $p$-sets and the $q$-sets by $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ respectively where $s \in S$ and $t \in T$ [See for products 9]. Here it is easy to see that $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t)$. Now $r \in \mathcal{P}(S) \otimes T$ is called a \textit{relation} from $(S, S)$ to $(T, T)$ if it satisfies

(i) $r \not\subseteq \overline{Q}_{(s,t)}$, $P_{s'} \not\subseteq Q_s$ \implies $r \not\subseteq \overline{Q}_{(s',t)}$, and

(ii) $r \not\subseteq \overline{Q}_{(s,t)}$ \implies $\exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

Dually, $R \in \mathcal{P}(S) \otimes T$ is called a \textit{correlation} from $(S, S)$ to $(T, T)$ if the following conditions hold:

(1) $\overline{P}_{(s,t)} \not\subseteq R$, $P_{s'} \not\subseteq Q_s$ \implies $\overline{P}_{(s',t)} \not\subseteq R$

(2) $\overline{P}_{(s,t)} \not\subseteq R$ \implies $\exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair $(r, R)$, where $r$ is a relation and $R$ a correlation from $(S, S)$ to $(T, T)$ is called a \textit{direlation} from $(S, S)$ to $(T, T)$.

Let $(f, F)$ be a direlation from $(S, S)$ to $(T, T)$. Then $(f, F)$ is called a \textit{difunction} from $(S, S)$ to $(T, T)$ if

(1) for $s, s' \in S$, $P_s \not\subseteq Q_{s'}$ \implies $\exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

(2) for $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F$ \implies $P_{t'} \not\subseteq Q_t$. 
Further, a difunction \((f, F)\) is

(i) \textit{surjective} if for \(t, t' \in T, P_t \not\subseteq Q_{t'} \implies \exists s \in S \text{ with } f \not\subseteq \overline{Q}_{(s,t')} \) and \(\overline{P}_{(s,t)} \not\subseteq F\).

(ii) \textit{injective} if for \(s, s' \in S \) and \(t \in T, f \not\subseteq \overline{Q}_{(s,t)} \) and \(\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}\).

For a convenient topological structure on textures, we consider that the open sets and closed sets are independent of each other. A \textit{ditopology} on a texture space \((S, S)\) is a pair \((\tau, \kappa)\) of subsets of \(S\), where the family of \textit{open sets} \(\tau\) satisfies

(1) \(S, \emptyset \in \tau\),
(2) \(G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau\) and
(3) \(G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau\),

and the family of \textit{closed sets} \(\kappa\) satisfies

(1) \(S, \emptyset \in \kappa\),
(2) \(K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa\) and
(3) \(K_i \in \kappa, i \in I \implies \bigcap_i K_i \in \kappa\).

In this respect, the closure and interior of a set \(A \in S\) in a ditopological texture space \((S, S, \tau, \kappa)\), is given by the equalities

\[ [A] = \bigcap \{ K \in \kappa \mid A \subseteq K \} \text{ and } ]A[ = \bigvee \{ G \in \tau \mid G \subseteq A \} \]

respectively.

Recall that ditopological texture spaces and bicontinuous difunctions form a category which is denoted by \textbf{dfDitop}. For some results on ditopologies, we refer \[3,14,15 \text{ and } 20\].

For the concepts on textures which are not explained here, see \[4–9\].

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2. Induced Subtextures

Substructures of textures are extensively discussed in \[8\] and \[12\]. One of the well known substructures of textures can be defined on elements of textures, that is if \((S, S)\) is a texture space and \(A \in S\), then the family \(S_A = \{ A \cap B \mid B \in S \} \) is a texture on \(A\). In this case, the pair \((A, S_A)\) is called the \textit{principal subtexture} of \((S, S)\). Since the principal subtexture is a natural counterpart in the theory of texture spaces, then most of the hereditary properties of textures can be obtained in a natural way. For instance, the complete biregularity can be given in terms of restricted difunctions.
which is given in [2]. Here, we don’t follow this line, and for any subset of $S$ we discuss the substructures of plain texture spaces since we will use it in products in the last chapter.

**Definition 2.1.** Let $(S, S)$ be a texture space and $A \subseteq S$. If the family $S_A = \{ A \cap B \mid B \in S \}$ is a texturing on $A$, then $S_A$ is called an induced structure on $A$, and $(A, S_A)$ is called an induced subtexture of $(S, S)$. If we define $\tau_A = \{ G \cap A \mid G \in \tau \}$ and $\kappa_A = \{ K \cap A \mid K \in \kappa \}$, then $(\tau_A, \kappa_A)$ is called the induced ditopology on $A$, and $(A, S_A, \tau_A, \kappa_A)$ is called an induced ditopological subtexture space of $(S, S, \tau, \kappa)$.

**Theorem 2.2.** Let $(S, S)$ be a plain texture space and $A \subseteq S$. Then the pair $(A, S_A)$ is an induced subtexture space of $(S, S)$ where $S_A = \{ U \cap A \mid U \in S \}$ is a texturing on $S$.

**Proof.** Since $(S, S)$ is plain, by [1, Theorem 2.1.4] arbitrary joins coincide with unions and so trivially, the family $S_A$ is a completely distributive lattice with respect to set inclusion. The other conditions are straightforward. □

**Example 2.3.** Consider the real ditopological texture space $(\mathbb{R}, \mathbb{R}, \tau, \kappa)$ where

$$\mathbb{R} = \{ (-\infty, a) \mid a \in \mathbb{R} \} \cup \{ (-\infty, a) \mid a \in \mathbb{R} \} \cup \{ \mathbb{R}, \emptyset \}$$

and $\tau = \{ (-\infty, a) \mid a \in \mathbb{R} \} \cup \{ \mathbb{R}, \emptyset \}, \{ (-\infty, a) \mid a \in \mathbb{R} \} \cup \{ \mathbb{R}, \emptyset \}$.

Since $(\mathbb{R}, \mathbb{R}, \tau, \kappa)$ is plain, then clearly, the pair $(\mathbb{N}, \mathbb{N})$ is an induced subtexture of $(\mathbb{R}, \mathbb{R})$ where $\mathbb{N} = \{ 0, 1, 2, \ldots \} \subseteq \mathbb{R}$ and

$$\mathbb{N} = \{ \mathbb{N} \cap A \mid A \in \mathbb{R} \} = \{ \{ 0, 1, 2, \ldots, a \} \mid a \in \mathbb{N} \} \cup \{ \mathbb{N}, \emptyset \}.$$ 

If we consider the induced ditopology $(\tau_{\mathbb{N}}, \kappa_{\mathbb{N}})$ on $(\mathbb{N}, \mathbb{N})$, then we may write $\tau_{\mathbb{N}} = \kappa_{\mathbb{N}} = \mathbb{N}$.

Since $(\mathbb{N}, \mathbb{N}, \tau_{\mathbb{N}}, \kappa_{\mathbb{N}})$ is a discrete codiscrete ditopological texture space, then it satisfies all separation axioms which will be mentioned in this paper.

Now let us denote the p-sets and the q-sets of an induced substructure $(A, S_A)$ by $P_s^A$ and $Q_s^A$ respectively for some $s \in A$.

**Lemma 2.4.** Let $(S, S)$ be a plain texture space and $(A, S_A)$ be an induced subtexture space of $(S, S)$. If $s \in A$, then $P_s^A \subseteq P_s \cap A$ and $Q_s \cap A \subseteq Q_s^A$. 
Proof. Since \((A, A)\) is an induced subtexture space, then for all \(s \in S\) we may write \(P_s \cap A \in S_A\) and so if \(s \in A\), then clearly, we have \(P_s^A \subseteq P_s \cap A\). For the second inclusion, let \(s \in A\) and \(t \in S\). Then we have \(A \cap P_t \in S_A\) and therefore, \(A \cap P_t = \bigcup \{P_r^A \mid r \in A \cap P_t\}\).

If \(s \notin A \cap P_t\), then by the first inclusion we may write that \(s \notin P_r^A\) where \(r \in A \cap P_t\). This implies that \(P_r^A \subseteq Q_s^A\), that is \(A \cap P_t \subseteq Q_s^A\).

On the other hand, \(Q_s \cap A \in S_A\) and then
\[
Q_s \cap A = \bigcup \{P_t \mid s \notin P_t\} \cap A = \bigcup \{A \cap P_t \mid s \notin P_t\} = \bigcup \{A \cap P_t \mid s \notin A \cap P_t\} \subseteq Q_s^A.
\]

\[\square\]

Example 2.5. [8] Consider the texture space \((S, S)\) where
\[S = (0, 1] \text{ and } S = \{(0, r] \mid r \in [0, 1]\}.
\]
We know that \((S, S)\) is not plain. Now let \(A = \{\frac{1}{2}, 1\}\) and consider the induced texture \(S_A = \{A, A, \{1/2\}\}\) on \(A\). Clearly, \(Q_1 = (0, 1]\) and \(Q_1^A = \{\frac{1}{2}\}\). However, \(Q_1 \cap A = A \nsubseteq \{\frac{1}{2}\} = Q_1^A\) and hence, we cannot remove the condition of “to be plain” of \((S, S)\) in Lemma 1.4.

3. Point separation properties

Now let us recall the following concepts.

Definition 3.1. [11] A ditopological texture space \((S, S, \tau, \kappa)\) is called
(a) \(T_0\) if for \(s, t \in S\), \(Q_s \nsubseteq Q_t \implies (\exists H \in \tau \cup \kappa)(P_s \nsubseteq H \nsubseteq Q_t)\).
(b) \(T_1\) if for \(s, t \in S\), \(Q_s \nsubseteq Q_t \implies (\exists K \in \kappa)(P_s \nsubseteq K \nsubseteq Q_t)\).
(c) co-\(T_1\) if for \(s, t \in S\), \(Q_s \nsubseteq Q_t \implies (\exists G \in \tau)(P_s \nsubseteq G \nsubseteq Q_t)\).
(d) bi-\(T_1\) if it is \(T_1\) and co-\(T_1\).
(e) \(T_2\) if \(T_0\) and \(R_1\)
(f) co-\(T_2\) if \(T_0\) and co-\(R_1\)
(g) bi-\(T_2\) if it is \(T_2\) and co-\(T_2\).

However much the point separation properties is given in terms of the p-sets and the q-sets, there are some characterizations of them which are independent from p-sets and q-sets [11] and here we prefer to use these characterizations to obtain the following results.
Theorem 3.2. Let \((S, \mathcal{S}, \tau, \kappa)\) be a ditopological plain texture space and \((A, \mathcal{S}_A, \tau_A, \kappa_A)\) be a ditopological induced subtexture space. If \((S, \mathcal{S}, \tau, \kappa)\) is bi-\(T_i\) for \(i = 0, 1, 2\), then \((A, \mathcal{S}_A, \tau_A, \kappa_A)\) is also bi-\(T_i\) for \(i = 0, 1, 2\).

Proof. Let \(B \in \mathcal{S}_A\). Then we have \(B = A \cap C\) for some \(C \in \mathcal{S}\). Then by [11, Theorem 4.7.(3)], we may write

\[ C = \bigvee_{j \in J_i} \bigcap_{i \in I_j} C^j_i \]

where \(C^j_i \in \tau \cup \kappa\). Since \((S, \mathcal{S})\) is plain, then

\[ B = A \cap C = A \cap \left( \bigcup_{j \in J_i} \bigcap_{i \in I_j} C^j_i \right) = \bigcup_{j \in J_i} \bigcap_{i \in I_j} (A \cap C^j_i) \]

where \(D^j_i = A \cap C^j_i \in \tau_A \cup \kappa_A\) and hence, \((A, \mathcal{S}_A, \tau_A, \kappa_A)\) is also \(T_0\). If \((S, \mathcal{S}, \tau, \kappa)\) is T_1 and \(B \in \mathcal{S}_A\), then by [11, Theorem 4.11 (1) (i)],

\[ C = \bigvee_{i \in I} F_i \]

where \(F_i \in \kappa, C \in \mathcal{S}\) and \(B = A \cap C\). Therefore,

\[ B = A \cap \left( \bigvee_{i \in I} F_i \right) = A \cap \left( \bigcup_{i \in I} F_i \right) = \bigcup_{i \in I} (A \cap F_i). \]

If \((S, \mathcal{S}, \tau, \kappa)\) is co–\(T_1\), using a similar argument and [11, Theorem 4.11 (2) (i)], it can be shown that \((A, \mathcal{S}_A)\) is also co–\(T_1\).

Let \((S, \mathcal{S}, \tau, \kappa)\) be bi–\(T_2\) and \(B \in \mathcal{S}_A\). Then we may write \(B = A \cap E\) for some \(E \in \mathcal{S}\). Since \((S, \mathcal{S}, \tau, \kappa)\) is bi–\(T_2\), then by [11, Theorem 4.17 (3)], there exists \(H^j_i \in \tau, K^j_i \in \kappa, i \in I, j \in J_i\) with \(H^j_i \subseteq K^j_i\) for all \(i, j\) and

\[ E = \bigvee_{i \in I} \bigcap_{j \in J_i} H^j_i = \bigvee_{i \in I} \bigcap_{j \in J_i} K^j_i \]

and so

\[ A \cap E = A \cap \left( \bigvee_{i \in I} \bigcap_{j \in J_i} H^j_i \right) = A \cap \left( \bigvee_{i \in I} \bigcap_{j \in J_i} K^j_i \right) \]

and since \((S, \mathcal{S}, \tau, \kappa_A)\) is plain, then

\[ A \cap E = \left( \bigcup_{i \in I} \bigcap_{j \in J_i} (A \cap H^j_i) \right) = \left( \bigcup_{i \in I} \bigcap_{j \in J_i} (A \cap K^j_i) \right). \]
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Now if we observe that $A \cap H^j_i \in \tau_A$ and $A \cap K^j_i \in \kappa_A$, then we find that $(A, S_A, \tau_A, \kappa_A)$ is also bi–$T_2$. □

Observing the above definitions, we may say that $T_0$ regular space is $T_3$, and $T_0$ co–regular space is co–$T_3$ and so $T_3$ and co–$T_3$ space is bi–$T_3$. Further, a $T_1$ normal space is $T_4$ and co–$T_1$ normal space is co–$T_4$.

4. COMPLETE REGULARITY AND NORMALITY

Definition 4.1. [11] A ditopological texture spaces $(S, S, \tau, \kappa)$ is

(a) completely regular if given $G \in \tau, G \not\subseteq Q_s$, there exists a bicontinuous difunction $(f, F) : (S, S, \tau, \kappa) \rightarrow (I, I, \tau_I, \kappa_I)$ satisfying $P_s \subseteq f^-P_0$ and $F^-Q_1 \subseteq G$.

(b) completely co–regular if given $K \in \kappa, P_s \not\subseteq K$, there exists a bicontinuous difunction $(f, F) : (S, S, \tau, \kappa) \rightarrow (I, I, \tau_I, \kappa_I)$ satisfying $K \subseteq f^-P_0$ and $F^-Q_1 \subseteq Q_s$.

(c) completely biregular if it is completely regular and completely co–regular.

(d) $T_{3\frac{1}{2}}$ if it is $T_0$ and completely biregular.

(e) normal if given $G \in \tau, K \in \kappa$ with $K \subseteq G$, there exist $H \in \tau$ and $M \in \kappa$ with $K \subseteq H \subseteq M \subseteq G$.

Now for general textures let us recall the following result.

Lemma 4.2. [10] Let $(S, S), (T, T)$ be textures and $\psi : S \rightarrow T$ be point function satisfying the following conditions:

(a) $s, s' \in S, P_s \not\subseteq Q_{s'} \implies P_{\psi(s)} \not\subseteq Q_{\psi(s')}$.

(b) $P_{\psi(s)} \not\subseteq B, B \in T \implies \exists s' \in S$ with $P_s \not\subseteq Q_{s'}$ for which $P_{\psi(s')} \not\subseteq B$.

(c) For $A \in T$ and $s \in S^\psi$ we have $A \not\subseteq Q_{\psi(s)} \implies A \not\subseteq Q_{\psi(u)}$ for some $P_u \not\subseteq Q_s$.

Then the difunction $(f_\psi, F_\psi)$ corresponding to $\psi$ satisfies the equalities

$$f_\psi = \bigvee \{P_{(s, \psi(s))} \mid s \in S\} \text{ and } G_\psi = \bigcap \{Q_{(s, \psi(s))} \mid s \in S^\psi\}.$$  

Further, $f_\psi^{-1}A = F_\psi^{-1}A = \psi^{-1}(A)$ for all $A \in T$.

As it is known that ditopological texture spaces and bicontinuous point functions satisfying the conditions (a) and (b) in Lemma 4.2
form a category which is denoted by \( \text{fDitop} \). Recall that if we get the objects as plain, then we obtain a category which is denoted by \( \text{fPDitop} \) whose morphisms are point functions satisfying the condition (a), and in this case note that the conditions (b)-(c) are automatically satisfied \([10]\).

Now we need the following lemma.

**Lemma 4.3.** Let \((S, S), (T, T)\) be texture spaces and \(\psi : S \to T\) be a point function satisfying the conditions (a)-(c) in Lemma 4.2. If \((A, S_A)\) is a plain induced subtexture space of \((S, S)\), then the restriction function \(\psi|_A : A \to T\) also satisfies the conditions (a)-(c).

**Proof.** Let \(s, s' \in A, P^A_s \nsubseteq Q^A_{s'}\). Then by Lemma 1.4, we have \(P_s \cap A \nsubseteq Q_{s'} \cap A\) and so we find \(P_s \nsubseteq Q_{s'}\). Since the function \(\psi\) satisfies the condition (a), then we may write \(P_{\psi(s)} \nsubseteq Q_{\psi(s')}\). Further, \(s, s' \in A\) gives that \(P_{\psi|_A(s)} \nsubseteq Q_{\psi|_A(s')}\). Therefore, restriction function \(\psi|_A : A \to T\) satisfies the condition (a). Since \(A\) is plain, then the conditions (b) and (c) are automatically satisfied for \(\psi|_A\). \(\square\)

The complete biregularity can be characterized in terms of point functions in the category of \(\text{fPDitop}\).

**Theorem 4.4.** Let \((S, S, \tau, \kappa) \in \text{ObfPDitop}\). Then we have the following.

(i) \((S, S, \tau, \kappa)\) is completely regular if and only if given \(G \in \tau, G \nsubseteq Q_s\), there exists a morphism \(\psi : S \to I\) in \(\text{fPDitop}\) satisfying \(\psi(P_s) = \{0\}\) and \(\psi(S \setminus G) = \{1\}\).

(ii) \((S, S, \tau, \kappa)\) is completely co–regular if and only if given \(K \in \kappa, P_s \nsubseteq K\), there exists a morphism \(\psi : S \to I\) in \(\text{fPDitop}\) satisfying \(\psi(S \setminus Q_s) = \{1\}\) and \(\psi(K) = \{0\}\).

**Proof.** (i) (\(\Leftarrow\)) Let \(G \in \tau, G \nsubseteq Q_s\). Then for a morphism \(\psi : S \to I\) in \(\text{fPDitop}\) we have \(\psi(P_s) = \{0\}\) and \(\psi(S \setminus G) = \{1\}\). Therefore, by Lemma 5.1, there exists a difunction \((f_\psi, F_\psi)\) corresponding to \(\psi\) satisfies the equalities

\[
f_\psi = \bigvee \{P_{s,\psi(s)} \mid s \in S\} \quad \text{and} \quad G_\psi = \bigcap \{Q_{s,\psi(s)} \mid s \in S\}.
\]

Since \(\psi\) is bicontinuous and \(f_\psi^- A = F_\psi^- A = \psi^{-1}(A)\) for all \(A \in T\), then \((f_\psi, F_\psi)\) is also bicontinuous. Suppose that \(P_s \nsubseteq f_\psi^- P_0\).
Choose a point \( s' \in S \) where \( P_s \not\subseteq Q_{s'} \) and \( P_{s'} \not\subseteq f^{-}_\psi P_0 \). Then there exists \( t \in \mathbb{I} \) such that \( f_\psi \not\subseteq \overline{Q}(s',t) \) and \( P_t \not\subseteq P_0 \). By definition of \( f_\psi \), we have \( P_{\psi(s')} \not\subseteq Q_t \). Since \( P_t \not\subseteq P_0 \), then \( P_{\psi(s')} \not\subseteq P_0 \), namely \( \psi(s') \not\subseteq P_0 \) and so \( \psi(s') \not= 0 \). However, \( \psi(P_s) = 0 \) and \( P_s \not\subseteq Q_{s'} \) implies that \( \psi(s') = 0 \) and this is a contradiction.

(\( \implies \):) Now suppose that \( f^{-}_\psi Q_1 \not\subseteq G \). Take a point \( s' \in S \) where \( f^{-}_\psi Q_1 \not\subseteq Q_{s'} \) and \( P_{s'} \not\subseteq G \). Then there exists a point \( t \in \mathbb{I} \) such that \( \overline{P}(s',t) \not\subseteq F_\psi \) and \( Q_1 \not\subseteq Q_t \). By definition of \( F_\psi \), we have \( \overline{P}(s',t) \not\subseteq \overline{Q}(r,\psi(r)) \) for some \( r \in S \). Therefore, \( P_t \not\subseteq Q_{\psi(s')} \) and so \( Q_1 \not\subseteq Q_t \), that is \( \psi(s') < 1 \). On the other hand, \( s' \in S \setminus G \) and so \( \psi(s') = 1 \) is an immediate contradiction.

(ii) (\( \iff \)) Now let \( P_s \not\subseteq K \in \kappa \). Suppose that \( K \not\subseteq f^{-}_\psi P_0 \). Take a point \( s' \in S \) where \( K \not\subseteq Q_{s'} \) and \( P_{s'} \not\subseteq f^{-}_\psi P_0 \). Then there is a point \( t \in \mathbb{I} \) such that \( f_\psi \not\subseteq \overline{Q}(s',t) \) and \( P_t \not\subseteq P_0 \). It is easy to check that \( P_{\psi(s')} \not\subseteq Q_t \) and so \( P_{\psi(s')} \not\subseteq P_0 \), that is \( \psi(s') \not= 0 \). However, \( s' \in K \) and hence, \( \psi(s') = 0 \) is a contradiction.

(\( \iff \)): Suppose that \( f^{-}_\psi Q_1 \not\subseteq Q_s \). Take a point \( s' \in S \) where \( f^{-}_\psi Q_1 \not\subseteq Q_{s'} \) and \( P_{s'} \not\subseteq Q_s \). Then there exists \( t \in \mathbb{I} \) such that \( \overline{P}(s',t) \not\subseteq F_\psi \) and \( Q_1 \not\subseteq Q_t \). Now we have \( P_t \not\subseteq Q_{\psi(s')} \) and hence, \( Q_{\psi(s')} \subseteq Q_t \). Since \( Q_1 \not\subseteq Q_{\psi(s')} \), then \( \psi(s') < 1 \). Since \( P_{s'} \not\subseteq Q_s \), then \( s' \in S \setminus Q_s \) and so \( \psi(s') = 1 \) is a contradiction.

\[ \square \]

**Theorem 4.5.** Let \((S, S, \tau, \kappa)\) be a ditopological plain texture space and \((A, S_A, \tau_A, \kappa_A)\) be a plain induced ditopological subtexture space.

(i) If \((S, S, \tau, \kappa)\) is completely regular, then \((A, S_A, \tau_A, \kappa_A)\) is also completely regular.

(ii) If \((S, S, \tau, \kappa)\) is completely co–regular, then \((A, S_A, \tau_A, \kappa_A)\) is also completely co–regular.

**Proof.** (i) Let \((S, S, \tau, \kappa)\) be completely regular and let \( G \not\subseteq Q_A^\downarrow \) where \( G \in \tau_A \). Since \( G = A \cap H \) for some \( H \in \tau \), then \( H \not\subseteq Q_s \). Further, since \( S \) is completely regular, then by Theorem 4.4, there exists a morphism \( \psi : S \to \mathbb{I} \) in \text{fPDitop} satisfying \( \psi(P_s) = \{0\} \) and \( \psi(S \setminus H) = \{1\} \). Then clearly, \( \psi|_A(P_A^\downarrow) = \{0\} \) and \( \psi|_A(A \setminus G) = \{1\} \). By Lemma 4.3, \( \psi|_A : A \to \mathbb{I} \) is also a morphism in \text{fPDitop} satisfying the conditions (a)-(c). Therefore, by Theorem 4.4, \((A, S_A, \tau_A, \kappa_A)\) is completely regular.

(ii) The proof is dual to (i) \[ \square \]
Corollary 4.6. Let \((S, \mathcal{S}, \tau, \kappa)\) be a bi\(-\)\(T_1\) normal ditopological plain texture space and \(A \in \mathcal{S}\). Then the plain ditopological induced subtexture space \((A, \mathcal{S}_A, \tau_A, \kappa_A)\) is completely biregular.

Proof. By Corollary 5.24 in [11], \((S, \mathcal{S}, \tau, \kappa)\) is completely biregular and so by Theorem 4.5, \((A, \mathcal{S}_A, \tau_A, \kappa_A)\) is also completely biregular. \(\square\)

5. PRODUCT TEXTURE SPACES

Now for \(i \in I\), let \((S_i, \mathcal{S}_i, \tau_i, \kappa_i)\) be a ditopological texture space. Consider the product ditopological texture space \((S, \mathcal{S}, \tau, \kappa)\) where 

\[
S = \prod_{i \in I} S_i
\]

Take a point \(s = (s_i)_{i \in I} \in S\). Let 

\[
D(s, j)_i = \begin{cases} 
S_j & \text{if } i = j \\
\{s_i\} & \text{otherwise}
\end{cases}
\]

and let \(D_j\) be the product texture on \(D(s, j)\).

Theorem 5.1. \((D(s, j), D_j)\) is an induced subtexture space of the product space \((S, \mathcal{S})\) where 

\[
S = \prod_{i \in I} S_i.
\]

Proof. Immediate. \(\square\)

Theorem 5.2. [13] (i) The function \(\varphi : S_j \to D(s, j)\) defined by 

\[
\varphi(a) = (a_i)_{i \in I}, a \in S_j
\]

satisfies the conditions (a)-(c) in Lemma 4.2. Further, the restriction \(\pi_j|_{D(s, j)} : D(s, j) \to S_j\) is inverse of \(\varphi\).

(ii) The mapping \(\varphi\) is a textural homeomorphism in \texttt{fDitop}.

Proof. (i) For (a) let \(s_j, s'_j \in S_j\) and \(P_{s_j} \not\subseteq Q_{s'_j}\). By [9, Proposition 1.3], 

\[
P_{\varphi(s_j)} = \prod_{i \in I} P_i,
\]

where \(P_i = \{s_i\}\) for \(i \neq j\) and \(Q_{\varphi(s'_j)} = \bigcup_{i \in I} E(i, Q_{s'_j})\). Since we have \(Q_{s'_j} = \emptyset\) for \(i \neq j\), we may write 

\[
Q_{\varphi(s'_j)} = E(j, Q_{s'_j})
\]

and so clearly, \(P_{\varphi(s_j)} \not\subseteq Q_{\varphi(s'_j)}\). Now let \(P_{\varphi(s_j)} \not\subseteq B, B \in D_j\). Choose a point \(r = (r_i)_{i \in I} \in D(s, j)\) where \(P_{\varphi(s_j)} \not\subseteq Q_r\) and \(P_r \not\subseteq B\). Then, \(P_{s_j} \not\subseteq Q_{r_j}\) and \(P_{\varphi(r_j)} \not\subseteq B\) and this verifies (b). For (c) let \(B \in D_j, s_j \in S_j^0\) and \(B \not\subseteq Q_{\psi(s_j)}\). Since \(B \in D_j\), then \(B = \prod_{i \in I} B_i\) where \(B_i = \{s_i\}\) for \(i \neq j\). Therefore, \(B_j \not\subseteq Q_{s_j}\).
and so for some \( u \in S_j \), we have \( B_j \not\subseteq Q_{u_j} \) and \( P_{u_j} \not\subseteq Q_{s_j} \) and so \( B \not\subseteq Q_{\varphi(u_j)} \). Clearly, \( \varphi \) is the inverse of it and the proof of (i) is complete. (ii) Clearly, \( \varphi \) is bijective and it is bicontinuous [14, Lemma 2.10]. Further, since the inverse \( \pi_j|_{D(s,j)} \) is also a projection function, it satisfies the conditions (a)-(c) in Lemma 4.2 and it is bicontinuous [10, Lemma 3.9].

**Theorem 5.3.** For the function \( \pi_j|_{D(s,j)} : D(s,j) \longrightarrow S_j \), the equalities
\[
f_\psi = \bigvee \{ \mathcal{T}_{(s,\psi(s))} \mid s \in S_j \} \quad \text{and} \quad F_\psi = \bigcap \{ \mathcal{Q}_{(s,\psi(s))} \mid s \in S_j \}
\]
define a dihomeomorphism \( (f_\psi, F_\psi) \) from \( D(s,j) \) to \( S_j \) where \( \psi = \pi_j|_{D(s,j)} \).

**Proof.** Since the function \( \psi \) is bijective, it is easy to see that corresponding difunction \( (f_\psi, F_\psi) \) is also bijective. Further, \( \pi_j|_{D(s,j)} \) is a textural homeomorphism in \( \mathfrak{fditop} \), and so in view of Theorem 2.8 in [2], \( (f_\psi, F_\psi) \) is a dihomeomorphism in \( \mathfrak{dfditop} \). □

**Theorem 5.4.** Let \( \{(S_i, S_i, \tau_i, \kappa_i) : i \in I\} \) be a family of non-empty ditopological plain texture spaces. If the product ditopological texture space \( (S, S, \tau, \kappa) \) is completely biregular, then the ditopological induced subtexture space \( (D(s,j), D_j, \tau_{D(s,j)}, \kappa_{D(s,j)}) \) is also completely biregular.

**Proof.** Since \( (D(s,j), D_j, \tau_{D(s,j)}, \kappa_{D(s,j)}) \) is plain, the proof is immediate by Theorem 4.5. □

**Theorem 5.5.** For \( i \in I \) let \( (S_i, S_i, \tau_i, \kappa_i) \) be non-empty ditopological plain texture spaces and \( (S, S, \tau, \kappa) \) be their product. Then \( (S, S, \tau, \kappa) \) is completely biregular if and only if \( (S_i, S_i, \tau_i, \kappa_i) \) is completely biregular for all \( i \in I \).

**Proof.** By the preceding theorem, the induced ditopological subtexture space \( (D(s,j), D_j, \tau_{D(s,j)}, \kappa_{D(s,j)}) \) is also completely biregular and since dihomemomorphisms preserves the complete biregularity [11, Proposition 5. 27], in view of Theorem 5.8, \( S_i \) is completely biregular for all \( i \in I \). The proof of the second part of the theorem is proved for general textures [11, Theorem 5.16]. □

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