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by

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SLICES ON THE BOUNDARY OF SCHOTTKY SPACE OF GENUS 2

RAQUEL ÁGUEDA

Abstract. Let $\mathcal{R}$ be the deformation space of free Kleinian groups generated by a parabolic and a loxodromic element, which correspond to representations into $PSL(2,\mathbb{C})$ of the fundamental group of a doubly cusped handlebody M whose boundary surface is a twice punctured torus. In this paper we show that this parameter space appears as the natural generalization of 1-complex dimensional slices which lie on its boundary: the Maskit embedding of a once punctured torus and the Riley slice of a four punctured sphere.

1. Introduction

A discrete subgroup Γ of $PSL\,(2,\mathbb{C}) = SL\,(2,\mathbb{C})/\pm Id$ is a *Kleinian group*. It acts as a group of isometries in the 3-dimensional hyperbolic space $\mathbb{H}^3$ and as a group of Möbius transformations (conformal automorphisms) on its boundary $\widehat{\mathbb{C}}$. The set $\Omega(\Gamma)$ is the *regular set* of Γ, the maximal subset of $\widehat{\mathbb{C}}$ where Γ acts properly discontinuously, and $\Lambda\,(\Gamma)$ is its complement, the *limit set*. A *marked Kleinian group* is a Kleinian group together with a set of generators.

Let M be a hyperbolic 3-manifold with boundary ∂M. A *simple loop* γ in ∂M is a closed curve with no self-intersections and it is *boundary parallel* if it is homotopic to a loop around a puncture. A simple loop is *essential* if it is neither null-homotopic nor boundary

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parallel in ∂M. A *curve system* $\underline{\gamma}$ is a set of disjoint and homotopically distinct essential simple loops. An essential simple loop γ in ∂M is *dividing* if by cutting the surface ∂M along γ we get two different surfaces and it is *non-dividing* if after the process of cutting along the curve we still obtain a unique surface.

A representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ of the fundamental group of M into $SL(2,\mathbb{C})$ is *discrete* if its image is a Kleinian group and *type-preserving* if it takes elements in $\pi_1(M)$ corresponding to boundary parallel loops in M into parabolic elements. The manifold M can be recovered as $\left(\mathbb{H}^3 \cup \Omega(\Gamma)\right)/\Gamma$ where $\Gamma = \rho(\pi_1(M))$. Free homotopy classes of loops in M are in one-to-one correspondence with conjugacy classes of elements of the fundamental group $\pi_1(M)$.

The *convex hull* $\mathcal{C}(\Gamma)$ of the limit set is the closure of the set of all geodesics in $\mathbb{H}^3$ whose endpoints lie in $\Lambda(\Gamma)$. This set is Γ-invariant. The quotient $\mathcal{C}(\Gamma)/\Gamma$ is called the *convex core* of M and is the smallest closed convex submanifold that contains all closed geodesics in $\mathbb{H}^3/\Gamma$. The boundary of the convex core of the hyperbolic manifold M is a pleated surface made up of pieces of hyperbolic planes which meet along a set of geodesics called the pleating lamination. This is an important object to study hyperbolic structures in the interior of the manifold M.

The general aim of our work is locating the deformation space $\mathcal{R}$ of free Kleinian groups Γ generated by a loxodromic and a parabolic element, which correspond to discrete, faithful and type-preserving representations into $SL(2,\mathbb{C})$ of the fundamental group of a doubly cusped handlebody M whose boundary surface is a twice punctured torus. Every such representation gives rise to a different geometric structure in the interior of M with the same underlying topology. This parameter space lies on the boundary of Schottky space of genus 2.

A *pleating variety* $\mathcal{P}\left(\underline{\gamma}\right)$ is the locus in $\mathcal{R}$ on which the convex hull boundary is bent (pleated) along a fixed geodesic lamination. We focus on the case in which the bending locus can be realized as curve systems on ∂M. The parameter space $\mathcal{R}$ can be computable if one can locate the position of these varieties. By the results of Young-Eun Choi and Caroline Series in [2], we know that the traces of the elements representing essential simple loops in $SL(2,\mathbb{C})$ are local parameters for $\mathcal{R}$ in a neighborhood of $\mathcal{P}\left(\underline{\gamma}\right)$ and global parameters when we restrict ourselves to $\mathcal{P}\left(\underline{\gamma}\right)$. The pleating variety is the non-singular locus on which the images of the curves in $\underline{\gamma}$ by a representation of $\pi_1(M)$ into $SL(2,\mathbb{C})$ are hyperbolic and therefore have real trace. Keeping the traces real is then a sufficient condition to stay in the pleating variety. We will eventually reach a boundary point when one of the elements representing a curve in the pleating lamination becomes parabolic (a curve has been pinched and ∂M is no longer homeomorphic to the twice punctured torus, but either to a four punctured sphere or the union of a once punctured torus and a triply punctured sphere).

We set up a suitable family of representations of $\pi_1(M)$, algebraically the free group on two generators, into $SL(2,\mathbb{C})$. The image of each representation is a Kleinian group Γ generated by a loxodromic and a parabolic element. We give such a representation a convenient parametrization so that Γ depends on two complex parameters X and Y related to the traces of certain loxodromic group elements in Γ. We define the parameter space $\mathcal{R}$ as the set of parameters $(X,Y)\in\mathbb{C}^2$ where ∂M is a twice punctured torus.

We give a combinatorial description of simple loops in the boundary ∂M by parameterizing the space of homotopy classes of essential simple loops in ∂M using the **p**-coordinates developed by Linda Keen, John R. Parker, and Caroline Series in [4]. The top term of the trace polynomial of group elements which are the image by representations $\rho:\pi_1(M)\to SL(2,\mathbb{C})$ of curves in ∂M representing homotopy classes in M only depends on the combinatorics of the simple loops in ∂M and can be expressed in terms of the top terms of the traces of some simple elements of Γ.

We work out two examples of 1-dimensional slices which lie on the boundary of our parameter space $\mathcal{R}$ by pinching a separating and a non-separating curve, respectively. In this way we introduce an extra parabolic element and show that our results generalize the 1-complex dimensional cases studied by Linda Keen and Caroline Series in [5] and [6]: the Maskit embedding of a once punctured torus and the Riley slice of a four punctured sphere. We find relations between the **p**-coordinates we have used in our parametrization and the ones Keen and Series used in their papers and show that our trace formula generalizes theirs.

Keen and Series introduced the theory of pleating varieties as a tool to study deformation spaces of Kleinian groups. The second author and collaborators are currently working to extend this theory for general cases. Here we present a 2-dimensional example.

The contents of this paper are part of the author's Ph.D. dissertation. I wish to thank Caroline Series for very fruitful discussions and her constant encouragement.

2. Preliminaries

2.1. **Group Generators.**

Let M be the manifold obtained from a genus-2 handlebody by pinching the non-dividing curve β_3 (see Figure 1). Our manifold M should be seen by the reader as sitting outside of the boundary surface ∂M: a twice punctured torus where its two punctures are identified. Let us consider the space of discrete, faithful and type-preserving representations $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ defined up to conjugation into $PSL(2,\mathbb{C})$. In the same figure, let us consider the set of curves $\{\beta_1, \alpha_1, \alpha_2\}$. We obtain a set of generators of $\pi_1(\partial M)$ and consider the homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$ induced by the inclusion map $i : \partial M \hookrightarrow M$. This homomorphism is not injective since the curve α_2 is homotopically trivial in M.

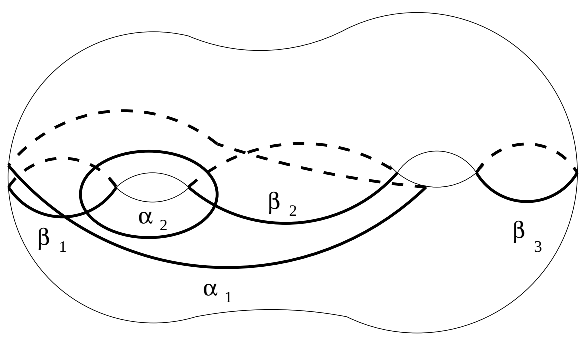


Figure 1

The set $\{\rho \circ i_*(\beta_1), \rho \circ i_*(\alpha_1), \rho \circ i_*(\alpha_2)\}$ is a set of generators of the Kleinian group $\rho(\pi_1(M))$. The image $\rho \circ i_*(\beta_1)$ of the essential simple closed curve β_1 is a loxodromic element which will be denoted by B_ρ, the element $A = \rho \circ i_*(\alpha_1)$ is parabolic since α_1

is homotopic in M to a boundary parallel loop around the identified punctures and $\rho \circ i_*(\alpha_2)$ is the identity matrix Id since α_2 bounds a disc in M (let us remind the reader that our manifold sits outside its boundary surface). So $\Gamma_\rho = \rho\left(\pi_1(M)\right)$ will be a free Kleinian group generated by two elements, one loxodromic and one parabolic.

We can normalize the generators of Γ_ρ so that the parabolic element A has fixed point ∞ and translation distance 1 and the loxodromic element B_ρ has fixed point 0, so

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_\rho = \begin{pmatrix} \cosh x + \sinh x & 0 \\ 2\left(\cosh y - \cosh x\right) & \cosh x - \sinh x \end{pmatrix}$$

where $(x, y) \in \mathbb{C}^2$, $x = \dfrac{\lambda_{b_\rho}}{2}$ and $y = \dfrac{\lambda_{ab_\rho}}{2}$, and λ_{b_ρ} and λ_{ab_ρ} are the complex translation lengths of B_ρ and AB_ρ, respectively.

The image of $\pi_1(M)$ by a given representation ρ is a Kleinian group $\Gamma_\rho = \langle A,\ B_\rho \rangle$ which depends on two complex parameters. If we want to make explicit the dependence of Γ_ρ on the complex parameters x and y we will write $\Gamma_{(x,y)} = \left\langle A, B_{(x,y)} \right\rangle$.

By Alhfors-Bers deformation theory the deformation space of geometrically finite groups Γ (Γ has a finite sided fundamental polyhedron) is completely described in terms of the deformation of $\Omega\left(\Gamma\right)/\Gamma$. In this context we will use the most convenient definition of the parameter space $\mathcal{R}$ in order to study how it sits inside $\mathbb{C}^2$: $\mathcal{R}$ corresponds to quasiconformal deformations of $\Gamma_{(x,y)}$ such that the boundary surface $\Omega\left(\Gamma_{(x,y)}\right)/\Gamma_{(x,y)}$ is a twice punctured torus

$$\mathcal{R} = \left\{ \begin{array}{c} (x,y) \in \mathbb{C}^2 \text{ such that } \Omega(\Gamma_{(x,y)})/\Gamma_{(x,y)} \\ \text{is a twice punctured torus} \end{array} \right\}.$$

For our convenience we will consider a new set of parameters $(X, Y) \in \mathbb{C}^2$ such that

$$\begin{aligned} X &= 2\cosh x = tr\left(B_\rho\right) \\ Y &= 2\left(\cosh y - \cosh x\right) = tr\left(AB_\rho\right) - tr\left(B_\rho\right). \end{aligned}$$

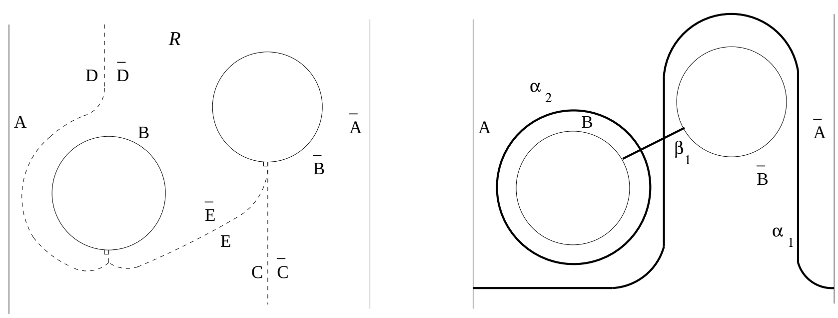
We can also write Y in terms of the trace of the commutator $[A, B_\rho] = AB_\rho A^{-1} B_\rho^{-1}$:

$$tr\left([A,\ B_\rho]\right) = \left(2\left(\cosh y - \cosh x\right)\right)^2 + 2 = Y^2 + 2.$$

2.2. **Homotopy classes of simple closed curves.**

An easy way to represent free homotopy classes of essential simple loops on the twice punctured torus ∂M was given in [4]. The authors introduced a model whereby every free homotopy class $[\gamma]$ of essential simple loops γ on ∂M can be assigned to a point in $(\mathbb{Z}_+ \cup \{0\})^2 \times \mathbb{Z}^2$, what we here call its **p**-coordinates and denote by $\mathbf{p}(\gamma) = (q_1(\gamma), \ q_2(\gamma), \ p_1(\gamma), \ p_2(\gamma))$. We refer the reader to this paper for full details. The **p**-coordinates determine and are determined by a representation of a simple loop γ as a weighted train track. We can extend this parametrization to homotopy classes of curve systems in ∂M. These coordinates are in fact a variant of the Dehn-Thurston coordinates with respect to the $\{\alpha_1, \ \alpha_2\}$-pants decomposition as described in [7] or [8].

A fundamental domain R for the action of Γ on $\widehat{\mathbb{C}}$ is given by the region inside the strip and outside the isometric circles drawn in Figure 2(a) where the label $W, \ W \in \{A, \ B\}$, on a side of R indicates that the side in question is carried by W to the side with label $\overline{W}$ (equivalently W^{-1}).



(a) (b)

FIGURE 2

In order to obtain a simply connected fundamental region we cut the fundamental domain R along the curves where the sides C and $\overline{C}$, and D and $\overline{D}$ respectively meet. The resulting domain is an octagonal simply connected region that lifts to a fundamental domain R' in the universal covering space of $\Omega(\Gamma)/\Gamma$, the unit disc Δ. In Figure 3(a) we draw the resulting fundamental domain R'.

We denote by Γ' the covering group generated by the side pairings A', B', C' and D'.

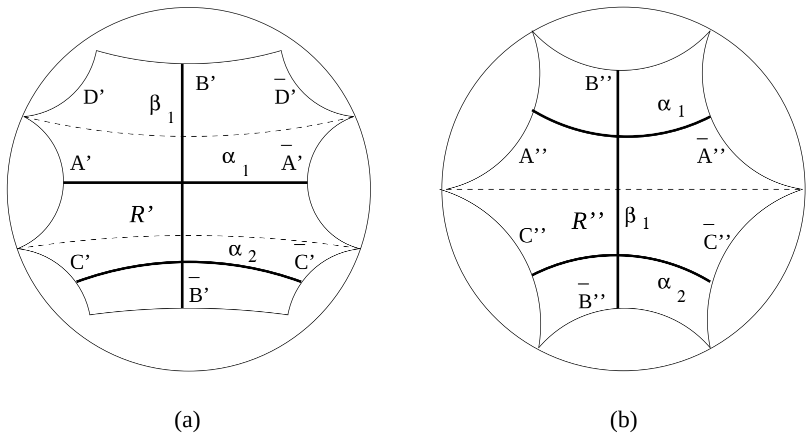


FIGURE 3

Now we cut R' into two different pieces R_1' (the rectangle in the center of R') and R_2' (given by the union of the two other rectangles, $R' - R_1'$) and consider a new fundamental domain $R'' = R_1' \cup B'(R_2')$ (see Figure 3(b)). We also represent in Figure 2(b) and Figure 3 a lift of some of the labeled curves drawn in Figure 1. The vertices of R'' lie in $\partial\Delta$ and project to punctures on $\Omega(\Gamma)/\Gamma$. We will denote by Γ'' the covering group generated by the side pairings A'', B'' and C''.

Since Γ is a free group on two generators A and B, the kernel of the homomorphism from Γ'' to Γ is not trivial. Recall that $\rho(\alpha_2) = Id$, the identity element in $SL(2,\mathbb{C})$, thus this homomorphism takes A'' to A, B'' to B and C'' to Id. A simple closed curve γ on ∂M appears on R'' as a collection of pairwise disjoint arcs running between the sides of R'', so that we can read off the element $\rho(\gamma)$ linking the endpoints of the arcs and, since $\rho(\alpha_2) = Id$, the labels C'' and $\overline{C''}$ are omitted. We obtain a word on $A^{\pm}$ and $B^{\pm}$ by performing cancellations in this product and all its cyclic permutations. Our parametrization is purely topological, thus we do not need to take account of the angles and therefore, for simplicity, we can represent the fundamental regions R' and R'' as rectangles divided into boxes. Let us represent every box R_i'' for $i = 1, 2$ as a square with a set of arcs joining every two sides and associated weightings $x_i,\, y_i,\ u_i \in \mathbb{Z}_+ \cup \{0\}$ as shown in Figure 4. Each of these weightings associated to a line is the number of strands joining two sides of the square R_i''. We can not have a vertical and a horizontal strand in the same box since otherwise the curve system would be self-intersecting, so the two boxes must necessarily be of one of the types shown in Figure 4.

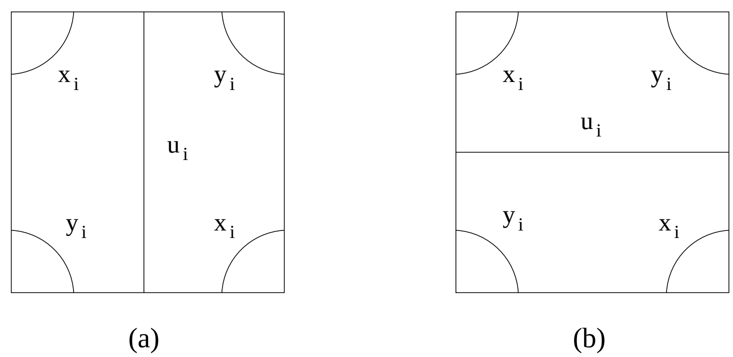


FIGURE 4

The numbers $x_i,\ y_i$ and u_i are positive integers that must fulfill certain basic conditions in order to represent a curve system. First, the number of strands that meet the sides W and $\overline{W}$ must necessarily be the same so that we have a collection of closed curves. Furthermore, these numbers $x_1,\ x_2,\ y_1$, and y_2 can not be all different from 0 since otherwise we would have a component which is boundary parallel.

We define the **p**-coordinates of a curve system $\underline{\gamma}$ from the numbers $x_i, y_i, u_i \in \mathbb{Z}_+ \cup \{0\}$ by $\mathbf{p}\left(\underline{\gamma}\right) = \left(q_1\left(\underline{\gamma}\right),\ q_2\left(\underline{\gamma}\right),\ p_1\left(\underline{\gamma}\right),\ p_2\left(\underline{\gamma}\right)\right)$. For the sake of readability we will write $\mathbf{p}\left(\underline{\gamma}\right) = (q_1,\ q_2,\ p_1,\ p_2)$ if this does not lead the reader to misunderstanding.

We consider two different cases: the **p**-coordinates for a pattern as the one in Figure 4(a) will be

$$\begin{aligned} q_i\left(\underline{\gamma}\right) &= u_i + |x_i - y_i| \\ p_i\left(\underline{\gamma}\right) &= \epsilon_i\, |x_i - y_i|\,, \end{aligned}$$

while for a pattern as in Figure 4(b)

$$\begin{aligned} q_i\left(\underline{\gamma}\right) &= |x_i - y_i| \\ p_i\left(\underline{\gamma}\right) &= \epsilon_i\left(u_i + |x_i - y_i|\right). \end{aligned}$$

where

$$\epsilon_i = \begin{cases} 1 \text{ if } x_i \geq y_i \\ -1 \text{ otherwise} \end{cases}.$$

We consider the set of curve systems in ∂M embedded into M. The induced homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$ between the fundamental groups is not a monomorphism, thus there are different elements in $\pi_1(\partial M)$ which are mapped to the same one in $\pi_1(M)$. In [1], we focus our work in determining when two distinct homotopy classes of curve systems in ∂M define the same homotopy class in M. We find a representative curve system $\underline{\gamma^0}$ in ∂M in every homotopy class in M, unique up to homotopy in ∂M. The **p**-coordinates of this representative element $\underline{\gamma^0}$ must fulfill the following conditions:

$$q_1\left(\underline{\gamma^0}\right) \leq q_2\left(\underline{\gamma^0}\right) \qquad \text{and} \qquad 0 \leq p_2\left(\underline{\gamma^0}\right) < q_2\left(\underline{\gamma^0}\right). \tag{2.1}$$

The key point of the proof consists in a"reducing" process based on performing certain homeomorphisms of ∂M which extend to maps that are homotopic to the identity in M (Dehn twists along curves which bound discs in M and a hyperelliptic involution) and studding their action on the set of curve systems. Given a curve system in ∂M and after performing one of these moves, the new curve system that we obtained is homotopically equivalent in M to the given one. The q_i-coordinate of a curve system is given by its intersection number with α_i. After performing one of these homeomorphisms, the q_1-coordinate of the new curve system must be smaller than the q_1-coordinate of the original one while the coordinate q_2 is a topological invariant and remains the same. After a finite number of steps we will have the desired result and the coordinates of the new curve system fulfill the condition $q_1 \leq q_2$. The condition $0 \leq p_2 < q_2$ is easily obtained after performing some Dehn twists along the curve α_2. Two curve systems will be homotopically equivalent in M if the representative elements obtained after performing this reducing process on each of them coincide.

2.3. **Trace polynomial.**

In [1], we find the first two terms of the trace polynomial of the group element that is the image by the representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ of an essential simple loop in ∂M representing a homotopy class in M. If two curves in ∂M define the same geodesic into M, then they are homotopic in M. Since the element in $SL(2,\mathbb{C})$ that represents all the curves in the same homotopy class in M is determined uniquely up to conjugation and cyclic permutation, then we use the representative element of the homotopy class whose **p**-coordinates fulfill the conditions that have been specified in (2.1). We calculate the trace polynomial in terms of these **p**-coordinates.

Theorem 2.1. *Let γ be a simple closed curve in ∂M with* **p***-coordinates $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ where $0 \neq q_1 \leq q_2$ and $0 \leq p_2 < q_2$. Let $\rho(\gamma)$ be the image of the curve γ by the representation ρ into $SL(2,\mathbb{C})$. Then the top term of the trace polynomial of the element $\rho(\gamma)$ is a homogenous polynomial of power q_2 in the complex variables X and Y of the following form.*

$$\begin{aligned} top(tr\,(\rho\,(\gamma))) \;\; = \;\; & \pm Y^{q_2-q_1} \\ & \cdot \left(X - sign\,(p_1)\left[\left|\frac{p_1}{q_1}\right|\right] Y\right)^{\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)q_1-|p_1|} \\ & \cdot \left(X - sign(p_1)\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)Y\right)^{|p_1|-\left[\left|\frac{p_1}{q_1}\right|\right]q_1} \end{aligned}$$

where $X = 2\cosh x$ and $Y = 2\,(\cosh y - \cosh x)$.

Furthermore, the term of power $q_2 - 1$ vanishes.

Let γ be a simple closed curve in ∂M with **p**-coordinates $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ which fulfill the theorem's conditions and $p_1 \geq 0$. The key point of the proof of this theorem consists in writing the image $\rho\,(\gamma)$ of the curve by the representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ as a chain of $\chi = \dfrac{q_2 - q_1}{2}$ blocks of products of matrices. Let us denote $\left[\dfrac{p_1}{q_1}\right]$, the integer part of $\dfrac{p_1}{q_1}$, by l. The group element $\rho\,(\gamma)$ can then be written

$$\prod_{i=1}^{\chi}\left(BA^{\pm}\prod_{j=1}^{k_i}\left(\overline{B}A^{l}\right)^{m_{ij}}\left(\overline{B}A^{l+1}\right)^{n_{ij}}\overline{B}A^{\pm}\prod_{j=1}^{k'_i}\left(B\overline{A}^{l}\right)^{m'_{ij}}\left(B\overline{A}^{l+1}\right)^{n'_{ij}}\right)$$

where

$$
\begin{aligned}
q_1 &= \sum_{i=1}^{\chi}\left(\sum_{j=1}^{k_i}(n_{ij}+m_{ij})+\sum_{j=1}^{k_i'}(n'_{ij}+m'_{ij})\right)\\
q_2 &= \sum_{i=1}^{\chi}\left(\sum_{j=1}^{k_i}(n_{ij}+m_{ij})+\sum_{j=1}^{k_i'}(n'_{ij}+m'_{ij})+2\right)\\
p_1 &= \sum_{i=1}^{\chi}\left(\sum_{j=1}^{k_i}\left(lm_{ij}+(l+1)\,n_{ij}\right)+\sum_{j=1}^{k_i'}\left(lm'_{ij}+(l+1)\,n'_{ij}\right)\right)
\end{aligned}
$$

and

$$m_{ij},\ n_{ij},\ m'_{ij},\ n'_{ij}\geq 0,$$

(see [4] for details). Our result is proved by studding the trace formula for one of such blocks and then by induction on the number of blocks χ ([1] contains full details). If $p_1 < 0$ the proof runs in a parallel way.

As a corollary we see that this formula can be interpreted in terms of the top terms of the traces of some basic elements of the group $\Gamma = \rho(\pi_1(M))$ so that

$$
\begin{aligned}
top(tr\,(\rho\,(\gamma))) = {}& \pm(top\,(tr\,[A,B]))^{\frac{q_2-q_1}{2}}\\
&\cdot\left(top\left(tr\left(\overline{B}A^{sign(p_1)\left[\left|\frac{p_1}{q_1}\right|\right]}\right)\right)\right)^{\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)q_1-|p_1|}\\
&\cdot\left(top\left(tr\left(\overline{B}A^{sign(p_1)\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)}\right)\right)\right)^{|p_1|-\left[\left|\frac{p_1}{q_1}\right|\right]q_1}.
\end{aligned}
$$

The exponents of this formula also have a nice geometrical meaning since we can divide the group element $\rho\,(\gamma)$ into:

(1) $\dfrac{q_2-q_1}{2}$ blocks of the type $[A,B]$ or $A\overline{B}AB$,

(2) $\left(\left[\left|\dfrac{p_1}{q_1}\right|\right]+1\right)q_1-|p_1|$ blocks of the type $\overline{B}A^{sign(p_1)\left[\left|\frac{p_1}{q_1}\right|\right]}$,

(3) $|p_1|-\left[\left|\dfrac{p_1}{q_1}\right|\right]q_1$ blocks of the type $\overline{B}A^{sign(p_1)\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)}$.

The top term of the trace polynomial for any group element of Γ is a good tool to study the asymptotic behavior of the pleating varieties.

3. Slices on the boundary of the parameter space

We study some 1-complex parameter slices which sit on the boundary of our parameter space $\mathcal{R}$ where

$$\mathcal{R} = \left\{ \begin{array}{c} (X,Y) \in \mathbb{C}^2 \text{ such that } \Omega(\Gamma_{(X,Y)})/\Gamma_{(X,Y)} \\ \text{is a twice punctured torus} \end{array} \right\}.$$

We illustrate the behavior of two topological possibilities by pinching a separating and a non-separating curve on the boundary surface ∂M. In this way we obtain an extra parabolic element and get the Maskit embedding of a once punctured torus and the Riley slice of a four punctured sphere, respectively. We find relations between the **p**-coordinates we have used in our parametrization and the ones Keen and Series used in [5] and [6] and see how the trace formula stated in Theorem 2.1 is a generalization of the formulae the authors obtained in the mentioned papers.

3.1. Maskit embedding of a once punctured torus.

We get a Maskit slice on the boundary of the parameter space by pinching a dividing curve γ in the surface ∂M which is equivalent to making parabolic the element which represents the curve into $SL(2,\mathbb{C})$. As a result of pinching one of these curves on the boundary surface we will get a new manifold whose interior is homeomorphic to the interior of M and has two different surfaces as boundary components: a once punctured torus and a triply punctured sphere. The 1-complex space we obtain on the boundary of $\mathcal{R}$ is the space of parameters

$$\mathcal{R}_M = \left\{ \begin{array}{c} a \in \mathbb{C} \text{ such that } \Omega\left(\Gamma_a\right)/\Gamma_a \text{ is a once punctured torus} \\ \text{and a triply puncture sphere} \end{array} \right\}.$$

Let us consider the curve β_4, (see Figure 5), represented into $SL(2,\mathbb{C})$ by the commutator $[A,B]$.

Lemma 3.1. *If the commutator $[A,B]$ is parabolic, then the parameter Y is $\pm 2i$.*

Proof. If the commutator $[A,B]$ becomes parabolic, then the trace of this element must be either 2 or -2.

$$tr\left([A,B]\right) = \left(2\left(\cosh y - \cosh x\right)\right)^2 + 2 = Y^2 + 2 = \pm 2.$$

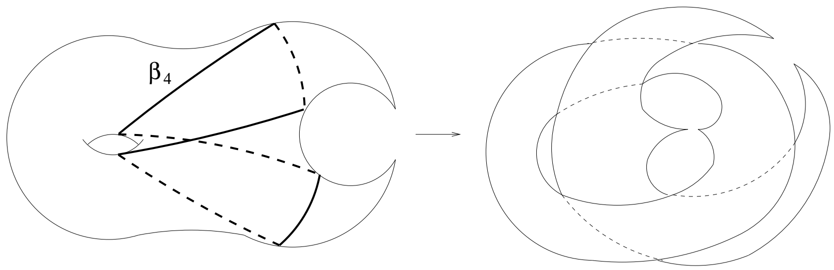


FIGURE 5

Since we have a discrete representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$, the group $\rho(\pi_1(M)) = \Gamma = \langle A, B\rangle$ must be a discrete non-elementary group, so Jørgensen inequality (see [3]) applies and

$$|tr\,(A) - 2| + |tr[A,B] - 2| \geq 1.$$

Thus, $tr\,([A,B]) \neq 2$ and therefore, the trace of the commutator $[A,B]$ must necessarily be -2 and

$$Y = 2\,(\cosh y - \cosh x) = \pm 2i. \qquad \square$$

Let us show that the top term of the trace polynomial which was obtained in [5] is equivalent to the one we have presented in Theorem 2.1 when the extra parabolic element has appeared and the parameter Y is $\pm 2i$.

In order to proceed we first need to find some correspondence between our parameter X and those used by the authors in their paper. Let us now consider a essential simple closed curve in $\partial M - \beta_4$. When we pinch β_4 the curve must lie on the once punctured torus component of the boundary of the manifold since all the curves in the triply punctured sphere are necessarily boundary parallel. We find the relation between its **p**-coordinates and the coordinates $(p,\ q)$ that Keen and Series used to parameterize simple closed curves on the once punctured torus.

Every free group generated by a loxodromic and a parabolic element such that the commutator of the two of them is parabolic is conjugated in $PSL(2,\mathbb{C})$ to a group generated by

$$\widetilde{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \widetilde{B} = \begin{pmatrix} -ia & -i \\ -i & 0 \end{pmatrix},$$

where $a \in \mathbb{C}$ (see [5]). Therefore, our generators A and B must be conjugate to $\widetilde{A}$ and $\widetilde{B}$, *i.e.*, there exists $G \in PSL(2, \mathbb{C})$ such that $GAG^{-1} = \widetilde{A}$ and $GBG^{-1} = \widetilde{B}$. Conjugate elements must have the same trace polynomial and this equality gives a relation between the parameters which appear in both pairs of generators.

$$\begin{aligned} tr\,(B) &= tr\left(\widetilde{B}\right) \Longleftrightarrow \\ X &= 2\cosh x = -ia. \end{aligned}$$

We can represent simple closed curves on a twice punctured torus in a rectangle with sides $A^{\pm}$ and $B^{\pm}$ paired as in Figure 6. The coordinates $(p,\ q)$ corresponding to this curve are obtained from this picture. The numbers $|p|$ and q are given by the number of strands meeting the A-side and the B-side, respectively. The rational number p/q corresponds to the slope of these strands where $|p|$ and q are coprime.

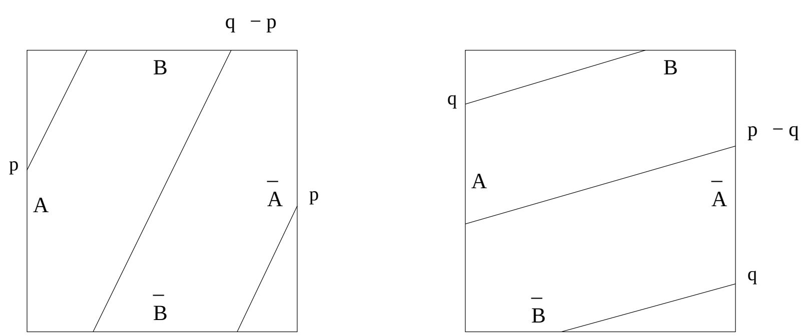


FIGURE 6

Theorem 3.2. *Let γ be a simple closed curve in $\partial M - \beta_4$ with* **p**-*coordinates* $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ *and let* $(p,\ q)$ *be the coordinates assigned to the same curve in the once punctured torus obtained by pinching the non-dividing curve β_4 in ∂M, then*

$$p = p_1 \quad \text{and} \quad q = q_1 = q_2.$$

Proof. It is easy to see that if γ is a simple closed curve which does not intersect β_4, then it must have **p**-coordinates $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ such that $q_1 = q_2$ and $p_2 = 0$ (see Figure 7 where we draw the possible diagrams when $p_1 \geq 0$. The curve γ is represented with a thin line while the curve β_4 is represented with a thick one). After pinching β_4, we can represent the curve into a rectangle–the way we also do in Figure 7–in order to find the relation between $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ and the $(p,\ q)$-coordinates. By comparing Figure 6 and Figure 7, we deduce that

$$p = p_1 \quad \text{and} \quad q = q_1 = q_2. \qquad \square$$

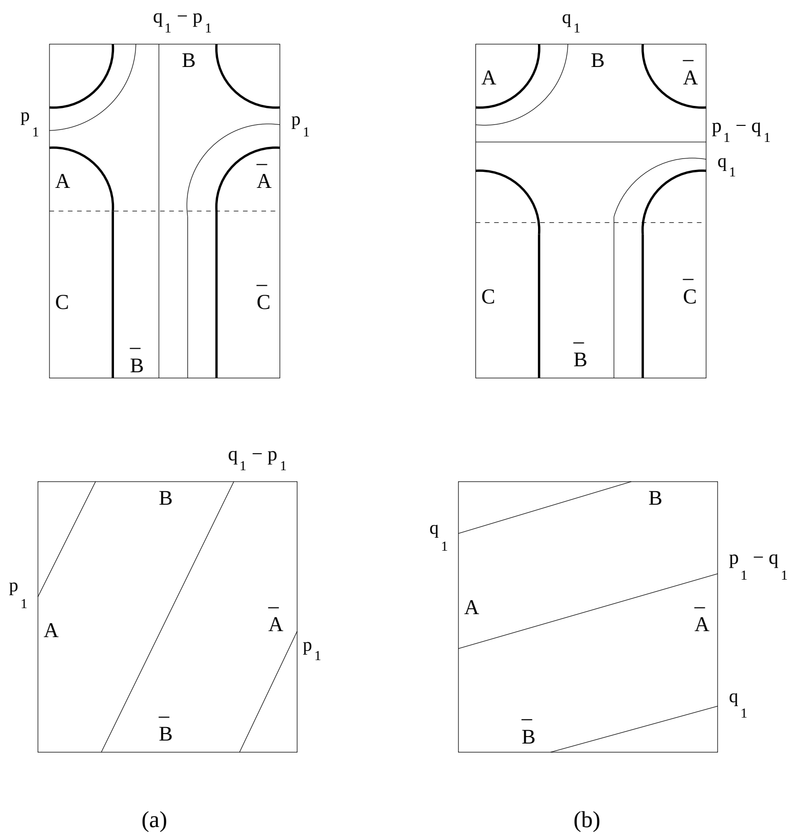


FIGURE 7

Corollary 3.3. *The top term of the trace polynomial of a group element in* $\left\langle \widetilde{A},\ \widetilde{B} \right\rangle$ *representing the homotopy class of a simple* *loop with coordinates* $(p,\ q) \in \mathbb{Z} \times \mathbb{Z}^+$ *which is given by* $(-ia)^q$ *(see* [5]*) coincides up to sign with the formula presented in Theorem 2.1.*

Proof. We have seen in Theorem 2.1 that the top term of the trace polynomial $tr\,(\rho\,(\gamma))$ has the following form:

$$\pm Y^{q_2-q_1} \cdot \left(X - sign\,(p_1)\left[\left|\frac{p_1}{q_1}\right|\right] Y\right)^{\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)q_1-|p_1|}$$
$$\cdot \left(X - sign(p_1)\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right) Y\right)^{|p_1|-\left[\left|\frac{p_1}{q_1}\right|\right]q_1} .$$

If $\gamma \in \partial M - \beta_4$, we know that $q_1 = q_2$, so that $q_1 - q_2 = 0$. Furthermore, we know that $Y = \pm 2i$ and the top term of the trace polynomial becomes

$$\pm \left(X \pm 2\left[\left|\frac{p_1}{q_1}\right|\right] i\right)^{\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right)q_1-|p_1|}$$
$$\left(X \pm 2\left(\left[\left|\frac{p_1}{q_1}\right|\right]+1\right) i\right)^{|p_1|-\left(\left[\left|\frac{p_1}{q_1}\right|\right]\right)q_1} .$$

When we go off to infinity, as X gets big enough,

$$top(tr\,(\rho\,(\gamma))) = \pm X^{q_1},$$

and by substituting the values of the **p**-coordinates by the corresponding to the coordinates $(p,\ q)$,

$$q_1 = q_2 = q,\ p_1 = p \text{ and } p_2 = 0,$$

and changing the parameter X by $-ia$, we show that our top term of the trace polynomial X^{q_1} becomes $(-ia)^q$, so that our formula coincides up to sign with the one obtained in [5]. □

3.2. **Riley slice of a four punctured sphere.**

Another special case on the boundary of the parameter space is obtained by pinching a non-dividing curve on the surface ∂M. This time we end up having a manifold with a unique boundary surface which is topologically a four punctured sphere.

The 1-complex slice we obtain on the boundary of $\mathcal{R}$ is now the deformation space

$$\mathcal{R}_R = \{b \in \mathbb{C} \text{ such that } \Omega\,(\Gamma_b)\,/\Gamma_b \text{ is four punctured sphere}\}\,.$$

Let us consider the manifold with boundary we obtain by pinching the curve β_1 (see Figure 8). The element representing β_1 in $SL(2,\mathbb{C})$ is B which is the extra parabolic we have in the group $\Gamma = \rho(\pi_1(M))$.

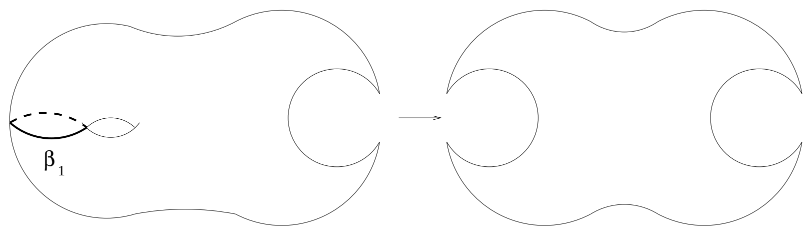


FIGURE 8

The element B becomes parabolic if and only if

$$tr(B) = 2\cosh x = X = \pm 2.$$

We again try to find the right correspondence between the parameters we are working with and the ones used in [6] and between the **p**-coordinates for a curve in $\partial M - \beta_1$ and the coordinates used in [6] to parameterize the set of simple closed curves in a four punctured sphere.

Any pair of non-commuting parabolic elements in $SL(2,\mathbb{C})$ can be normalized so that they have fixed points ∞ and 0 and the first one has translation length 1. We consider a new set of generators $\widetilde{A}$ and $\widetilde{B}$ for the group $\Gamma = \rho(\pi_1(M))$ generated by two non-commuting elements, which after normalizing are

$$\widetilde{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \widetilde{B} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

for $b \in \mathbb{C}$ (see [6]). As in the Maskit slice we will get a relation between the parameters by looking at the traces of some elements.

Neither $tr(A)$ nor $tr(B)$ and their conjugate elements help anyhow to deduce the relation since they are both parabolic, so we compare the traces of some other elements: AB and $\widetilde{A}\widetilde{B}$. These elements are conjugate and therefore their traces are necessarily the same one. Then

$$Y + X = tr(AB) = tr(\widetilde{A}\widetilde{B}) = 2 + b.$$

As $X = \pm 2$, Y must be either b or $b + 4$.

A fundamental polygon for the action of the group is given by a hexagon that we represent in Figure 9. The coordinates of a curve which lies in $\partial M - \beta_1$ used in [6] come from these patterns corresponding to the various cases that might appear. The number q is given by the number of strands meeting the A-side or, equivalently the B-side, and p corresponds to the number of strands meeting the C-side.

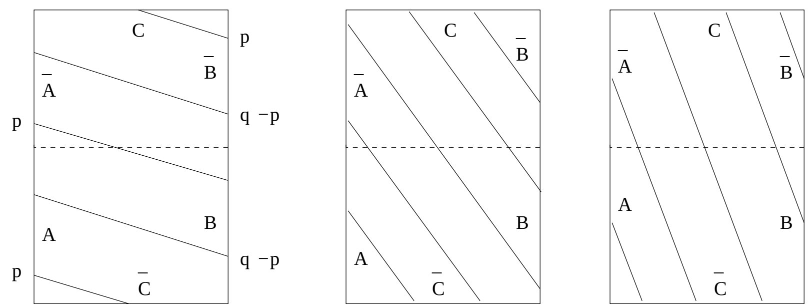


FIGURE 9

Again we check that the top term of the trace polynomial which was obtained in [6] is equivalent to the one we have just obtained when $X = \pm 2$ and first we need to find the correspondence between the **p**-coordinates of a curve γ in $\partial M - \beta_1$ and its $(p,\ q)$-coordinates.

Theorem 3.4. *Let γ be a simple closed curve in $\partial M - \beta_1$ with* **p**-*coordinates $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$ and let $(p,\ q)$ the coordinates assigned to the same curve in the four punctured sphere obtained by pinching the non-dividing curve β_1 in ∂M, then*

$$\begin{aligned} p &= \frac{q_2 - p_1}{2} \\ q &= q_2. \end{aligned}$$

Proof. Let us work out the correspondence between the **p**-coordinates of the curve γ and the (p, q) that describe a simple closed curve in a four punctured sphere as it appears in [6]. We first draw the possible patterns we can obtain to represent curves in $\partial M - \beta_1$. Let $\gamma \in \partial M - \beta_1$ be any such curve with **p**-coordinates $\mathbf{p}(\gamma) = (q_1,\ q_2,\ p_1,\ p_2)$, then $q_1 = |p_1|$ (see Figure 10 which corresponds to curves γ such that $p_1 = x_1 - y_1 \geq 0$. The curve γ is represented with a thin line, while the curve β_1 is represented with a thick one.)

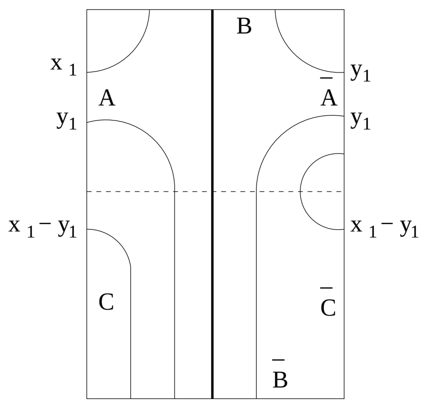


FIGURE 10

In Figure 11 we first study the action of a positive Dehn twist around the curve α_2 on the curve γ which intersects it. This action is given by a clockwise rotation, while a negative Dehn twist is given by a counterclockwise one. Observe that a Dehn twist acts linearly on the p_2-coordinate of the curve (this coordinate does not appear in the top term of the trace polynomial presented in Theorem 2.1), while the rest of the coordinates remain invariant.

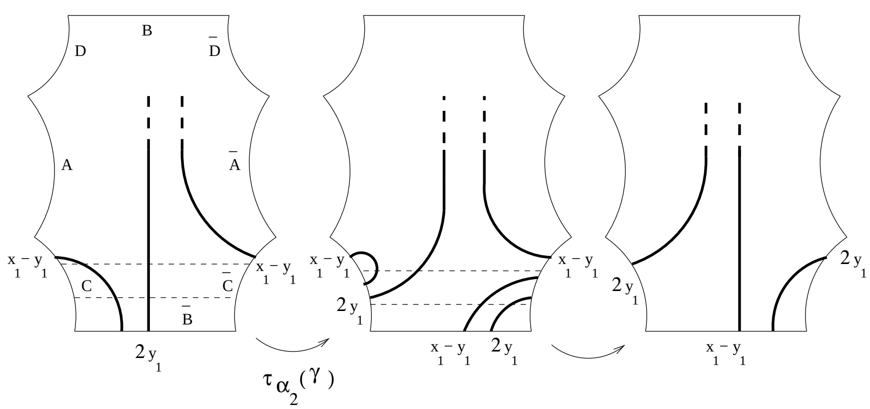


FIGURE 11

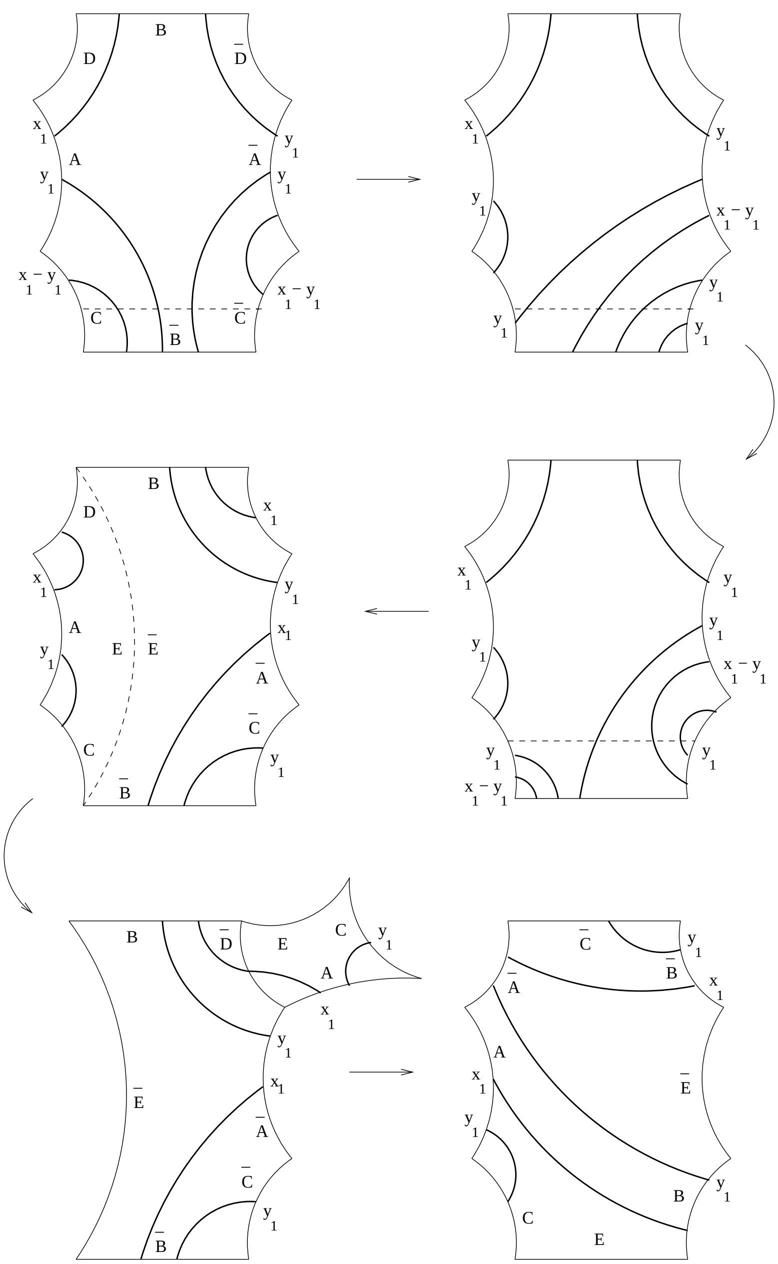


Figure 12

In Figure 12, we draw a chain of diagrams, starting with the diagram corresponding to a curve $\gamma \in \partial M - \beta_1$ into the simply connected fundamental polygon shown in Figure 3. In the first two steps, we compose a positive and a negative Dehn twist, both along α_2. The action is trivial on our curve. In the following picture, by cutting along the discontinuous curve and pasting back along B, we obtain a new fundamental polygon where our curve is represented. Finally we cut along $\overline{E}$ and paste back along $\overline{D}$.

All along this process we keep track of the number of strands that join the different sides of the polygons.

Observe that our curve does not cut neither of the sides E and $\overline{E}$ of our new fundamental polygon, so we can now represent the curve into a hexagon by omitting the E and $\overline{E}$-sides and in the corresponding 2-boxed rectangle (see Figure 13) and are finally ready

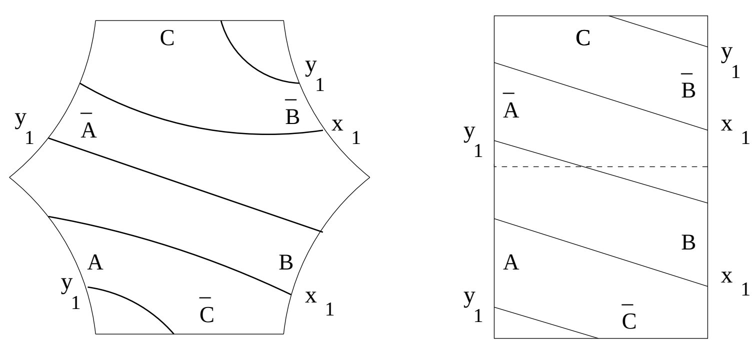


FIGURE 13

to write down the correspondence between the **p**-coordinates that we can write in terms of the numbers x_1 and y_1,

$$\begin{aligned} q_1 &= |x_1 - y_1| \\ p_1 &= x_1 - y_1 \\ q_2 &= x_1 + y_1, \end{aligned}$$

and the coordinates $(p,\ q)$ that we can associate to a simple closed curve in the four punctured sphere which has been obtained by pinching β_1. Remember that the numbers p and q are given by the number of strands ending up in the C-side and the A-side, respectively. So we have

$$\begin{aligned} p &= \frac{q_2 - p_1}{2} \\ q &= q_2. \end{aligned}$$

The proof for the case where $p_1 < 0$ runs in a parallel way. □

Corollary 3.5. *The top term of the trace polynomial of a group element in* $\left\langle \widetilde{A},\ \widetilde{B} \right\rangle$ *representing the homotopy class of a simple closed curve with coordinates* $(p,\ q)$ *which is given by* $(-1)^{p-q-1}\, b^q$ *(see* [6]*) coincides up to sign with the formula obtained in Theorem 2.1.*

Proof. Let us remind the reader that $\gamma \in \partial M - \beta_1$; thus, $q_1 = |p_1|$, and since $\rho\,(\beta_1)$ is parabolic, $X = \pm 2$. If $q_1 \neq 0$, by using Theorem 2.1, the top term of the trace polynomial $tr\,(\rho\,(\gamma))$ will be

$$\begin{aligned} & \pm Y^{q_2 - q_1} \cdot \left(X - sign\,(p_1) \left[\left| \frac{p_1}{q_1} \right| \right] Y \right)^{\left(\left[\left| \frac{p_1}{q_1} \right| \right] + 1 \right) q_1 - |p_1|} \\ & \cdot \left(X - sign(p_1) \left(\left[\left| \frac{p_1}{q_1} \right| \right] + 1 \right) Y \right)^{|p_1| - \left[\left| \frac{p_1}{q_1} \right| \right] q_1} \\ = \; & \pm Y^{q_2 - q_1} \, (\pm 2 \pm Y)^{2q_1 - |p_1|} \, (\pm 2 \pm 2Y)^{|p_1| - q_1} . \end{aligned}$$

When we go off to infinity, as our parameter Y gets big enough, we will have

$$top(tr\,(\rho\,(\gamma))) = \pm Y^{q_2}.$$

We have seen that the q_2-coordinate corresponding to any curve γ which does not intersect β_1 in the twice punctured torus is equivalent to the q-coordinate given in [6] for the same curve in the four punctured sphere we get by pinching β_1.

By substituting the parameter Y by b (or equivalently by $b + 4$), the top term of the trace polynomial becomes b^q. In [6], Keen and Series obtained the formula $(-1)^{p-q-1}\, b^q$ for the top term of the trace polynomial of the element $\rho\,(\gamma)$ with coordinates (p, q) which is, up to sign, equal to the one stated in Theorem 2.1.

The case $q_1(\gamma) = 0$ corresponds to the commutator $\rho\,(\gamma) = [A, B]$. We trivially see that the two formulae also coincide for this particular case. □

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