BOOLEAN ALGEBRAS AND
LOW SEPARATION AXIOMS

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Abstract. Let $B(\tau)$ be the smallest complete Boolean algebra containing the topology $\tau$. We present results of the following type: $\tau$ satisfies certain separation axioms if and only if $B(\tau)$ is equal to (naturally defined) subcollections of $B(\tau)$. Examples of such collections are the kerneled sets, the $\lambda$-closed sets, and the $\tau$-locally closed sets (where $\tau$ is the smallest Alexandroff topology containing $\tau$).

1. Introduction

Let $\tau$ be a topology over a set $X$ and let $\tau^*$ be the $\tau$-closed sets. By $B(\tau)$ we denote the smallest complete Boolean algebra containing $\tau$. In this paper we show that several low separation axioms (i.e., at most $T_2$) are characterized by properties of $B(\tau)$. In order to state our results we need to introduce some natural subclasses of $B(\tau)$.

A topology is said to be an Alexandroff topology if it is closed under arbitrary intersections. Juris Steprāns and Stephen Watson [13] attributed this notion to both Alexandroff and Tucker, and thus called them AT topologies. We will use their notation here. This class of topologies plays an important role in the study of low separation axioms. Notice that only the Alexandroff $T_1$ topology is the discrete topology.

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We denote by \( \tau \) the smallest Alexandroff topology containing \( \tau \). Let \( CO(\tau) \) be the collection of \( \tau \)-clopen subsets of \( X \). By \( \lambda_\tau^* \) we denote the collection of sets of the form \( K \cap C \) where \( K \in \tau \) and \( C \) is \( \tau \)-closed (i.e., the collection of \( \lambda \)-closed sets [1]), and by \( CX(\tau) \) the collection of \( \tau \)-locally closed sets (i.e., sets of the form \( K \cap S \) where \( K \in \tau \) and \( S \in \tau^\ast \)).

Let \( \tau(x) \) be the collection of open sets containing \( x \in X \). The \( \theta \)-closure of a subset \( A \) of \( X \) is defined in [2] as \( cl_\theta(A) = \{ x \in X : cl(V) \cap A \neq \emptyset \text{ for all } V \in \tau(x) \} \), where \( cl(V) \) is the \( \tau \)-closure operator on \( X \). Observe that \( cl_\theta(A) \) is the intersection of the closure of all open sets containing \( A \); hence, it is a closed set containing \( Ker(A) \). In particular, for any \( x, y \in X \), \( x \in cl_\theta(y) \) iff \( y \in cl_\theta(x) \) iff \( V \cap W \neq \emptyset \forall V \in \tau(x) \) and \( \forall W \in \tau(y) \). A set \( A \) is said to be \( \theta \)-closed if \( cl_\theta(A) = A \), equivalently if for all \( x \notin A \), there are disjoint open sets \( V \) and \( W \) such that \( x \in V \) and \( A \subseteq W \). The set \( cl_\theta(A) \) is not in general \( \theta \)-closed. Complements of \( \theta \)-closed sets are called \( \theta \)-open. Thus, \( A \) is \( \theta \)-open iff for each \( x \in A \) there is \( V \in \tau(x) \) such that \( cl(V) \subseteq A \). The family of all \( \theta \)-open sets forms a topology \( \tau_\theta \) on \( X \) which is clearly weaker than \( \tau \). Denoting by \( cl_{\tau_\theta}(A) \) the closure of \( A \) in the \( \tau_\theta \) topology, it is straightforward to prove that \( cl(A) \subseteq cl_\theta(A) \subseteq cl_{\tau_\theta}(A) \), for any \( A \subseteq X \). Since any \( \theta \)-open set can be written as a union of \( \tau \)-closed sets, then \( \tau_\theta \subseteq CO(\tau) \). Moreover, \( \tau \) and \( \tau_\theta \) have the same clopen sets. The following inclusion holds for any topology \( \tau \).

\[
CO(\tau) \subseteq \tau_\theta \subseteq \mathbb{B}(\tau_\theta) \subseteq CO(\tau) \\
CO(\tau) \subseteq \tau \subseteq \lambda_\tau^* \subseteq CX(\tau) \subseteq \mathbb{B}(\tau).
\]

Let \( \mathcal{A}(\tau) \) denote any of these subclasses of \( \mathbb{B}(\tau) \). Several low separation axioms are characterized by the fact that \( \mathcal{A}(\tau) = \mathcal{P}(X) \) (the power set of \( X \)). For instance, \( \tau \) is \( T_{1/2} \) iff every set is \( \lambda \)-closed [1], and \( \tau \) is \( T_{1/4} \) iff every set is \( \tau \)-locally closed [4]. In this paper we put these facts in a general framework. Instead of requiring that \( \mathcal{A}(\tau) = \mathcal{P}(X) \), we ask only that \( \mathcal{A}(\tau) \) is the complete Boolean algebra \( \mathbb{B}(\tau) \) and show that this requirement corresponds to a separation axiom. We use axioms \( S_i \) such that \( \tau \) is \( T_i \) iff it is \( S_i \) and \( T_0 \), for \( i = 1/4, 1/2, 1, \) or \( 2 \). An example of our results is that \( \tau \) is \( S_{1/4} \) iff \( CX(\tau) = \mathbb{B}(\tau) \).
The original motivation for this paper was [10], where results of this type were shown for countable spaces (in this case $\mathcal{B}(\tau)$ is the $\sigma$-algebra of Borel sets). Independently, Boolean algebras were used in [4] to analyze some separation axioms.

2. Terminology and preliminaries

If $\mathcal{A}$ is a collection of subsets of $X$, we will denote by $\mathcal{A}^*$ the collection $\{X \setminus A : A \in \mathcal{A}\}$. The kernel of a set $A \subseteq X$, denoted by $\ker(A)$, is the intersection of all open sets containing $A$. It is easy to check that $A \subseteq \ker(A)$ and that $\ker(\ker(A)) = \ker(A)$ for any $A \subseteq X$. Moreover, if $A \subseteq B$ then $\ker(A) \subseteq \ker(B)$. For any $x \in X$, we denote $\ker(\{x\}) = \ker(x)$. It is obvious that $x \in \cl(y)$ iff $y \in \ker(x)$. A set $A$ is said to be $\tau$-kerneled (or just kerneled) if $A = \ker(A)$. Equivalently, $A$ is kerneled iff $A = \bigcup_{x \in A} \ker(x)$. Kerneled sets are also called $\Lambda$-sets [11]. The family of all kerneled subsets of $X$ is closed under arbitrary unions and intersections, so it is an AT topology. Moreover, it coincides with $\overline{\tau}$. In fact, since every open set is kerneled and $\tau$ is the smallest AT topology containing $\tau$, then every member of $\overline{\tau}$ is kerneled. On the other hand, since $\overline{\tau}$ is closed under arbitrary intersections and it contains $\tau$, then every kerneled set belongs to $\overline{\tau}$.

A set $A \subseteq X$ is said to be $\tau$-saturated (or just saturated) if it contains the closure of all its points. It is clear that every closed set is saturated, and the concepts of saturated and closed coincide on finite sets. Notice that a set is saturated iff its complement is kerneled. Thus, the family $\overline{\tau}^*$ of the $\tau$-closed sets is precisely the family of the saturated sets.

If $A$ is the intersection of a kerneled set and a saturated set, then $A$ is said to be $\lambda$-closed [1]. Equivalently, $A$ is $\lambda$-closed iff $A = \ker(A) \cap \cl(A)$. The set $\ker(A) \cap \cl(A)$ is denoted in [1] by $\cl_{\lambda}(A)$. Thus, $A$ is $\lambda$-closed iff $\cl_{\lambda}(A) = A$. The operator $\cl_{\lambda}$ is monotone and idempotent. The complement of a $\lambda$-closed set is called $\lambda$-open. The collection of all $\lambda$-open sets is denoted by $\lambda^*_\tau$ and it is clear that a set is $\lambda$-open iff it is the union of an open set and a saturated set. The family $\lambda^*_\tau$ is closed under arbitrary unions, but it is not in general closed under finite intersection (see, for example, Example 4.3).
Each topology $\tau$ is associated with the following binary relation.

$$x \preceq_\tau y \iff x \in cl(y)$$

The relation $\preceq_\tau$ is transitive and reflexive (but in general, it is not antisymmetric) and is called the specialization preorder of $\tau$. An AT topology $\tau$ is uniquely determined by its associated preorder $\preceq_\tau$ in the sense that a set $V$ is $\tau$-open iff $V$ contains all $y$ such that $x \preceq_\tau y$ for all $x \in V$ (for a general presentation of AT topologies see [8]). Since $cl_\tau(x) = cl_\rho(x)$ for any $x \in X$, then $\preceq_\tau = \preceq_\rho$ iff $\tau = \rho$ for any pair of topologies $\tau$ and $\rho$ on $X$.

A set $A \subseteq X$ is said to be $\tau$-convex (or just convex) if the following holds: For all $x, y, z \in X$, if $x, z \in A$ and $x \preceq_\tau y \preceq_\tau z$, then $y \in A$, that is to say, $A$ is convex with respect to the preorder $\preceq_\tau$ of $\tau$. The collection of convex sets coincides with $CX(\tau)$. In fact, let $K \in \tau$ and $S \in \tau^*$, and let $x, z \in K \cap S$. If $x \preceq_\tau y \preceq_\tau z$, then $y \in \ker(x) \subseteq K$ and $y \in cl(z) \subseteq S$. Thus, $y \in K \cap S$, which shows that $K \cap S$ is convex. On the other hand, let $A$ be a convex set. Take $K = \bigcup_{x \in A} \ker(x)$ and $S = \bigcup_{x \in A} cl(x)$. It is clear that $A \subseteq K \cap S$. If $y \in K \cap S$, then $y \in \ker(x) \cap cl(z)$ for some $x, z \in A$ and thus, $x \preceq_\tau y \preceq_\tau z$. Therefore, $y \in A$.

3. Main results

In this section we prove the main result of this paper which is a characterization of some low separation axioms in terms of the Boolean algebra $\mathbb{B}(\tau)$. First, we introduce an equivalence relation over $(X, \tau)$:

$$(x, y) \in E_\tau \iff cl(x) = cl(y) \iff \ker(x) = \ker(y).$$

We denote the $E_\tau$-equivalence classes by $[x]_\tau$ for $x \in X$. We will show that they are the atoms of $\mathbb{B}(\tau)$. Note that

$$[x]_\tau = \{y \in X : x \preceq_\tau y \preceq_\tau x\}.$$

Now we introduce the separation axioms we are going to use in this article.

**Definition 3.1.** Let $\tau$ be a topology on $X$. Then $\tau$ is said to be

- \(T_0\) If for all $x, y \in X$ with $x \neq y$, there is an open set containing $x$ or $y$ but not both.
- \(T_{1/4}\) If every point of $X$ is closed or kernelled.
(T\(1/2\)) If every point of \(X\) is open or closed.

(T\(1\)) If every point of \(X\) is closed.

(T\(2\)) If for all \(x, y \in X\) with \(x \neq y\) there are disjoint open sets \(U \in \tau(x)\) and \(V \in \tau(y)\).

(S\(1/4\)) If \([x]_\tau\) is closed or kerneled for every \(x \in X\).

(S\(1/2\)) If \([x]_\tau\) is closed or open for every \(x \in X\).

(S\(1\)) If for all \(x, y \in X\), \(x \in \text{cl}(y)\) iff \(y \in \text{cl}(x)\), i.e. the closure of points forms a partition of \(X\).

(S\(2\)) If for all \(x, y \in X\) with \(\text{cl}(x) \neq \text{cl}(y)\), there are disjoint open sets \(U \in \tau(x)\) and \(V \in \tau(y)\) such that \(\text{cl}(x) \subseteq U\) and \(\text{cl}(y) \subseteq V\).

We prove in Theorem 3.11 that the separation axioms \(S_i\), for \(i = 2, 1, 1/2, 1/4\), can be characterized in terms of \(B(\tau)\):

\[
\begin{align*}
\tau \text{ is } S_2 & \iff B(\tau) = B(\tau) \\
\tau \text{ is } S_1 & \iff B(\tau) = \tau \\
\tau \text{ is } S_{1/2} & \iff B(\tau) = \lambda^*_\tau \\
\tau \text{ is } S_{1/4} & \iff B(\tau) = CX(\tau)
\end{align*}
\]

The most common notations for \(S_1\) and \(S_2\) are \(R_0\) and \(R_1\), respectively. However, the notation we use is more uniform (it is taken from [5]). The separation axioms \(R_0\) and \(R_1\) were introduced by N. A. Shanin [12] and A. S. Davis [7], respectively. To the best of our knowledge, the axioms \(S_{1/2}\) and \(S_{1/4}\) are introduced in the literature for the first time in this paper.

It is easy to verify that \(S_2 \Rightarrow S_1\) and that \(S_1\) is equivalent to asking that \([x]_\tau\) is closed for every \(x \in X\). Thus, \(S_2 \Rightarrow S_1 \Rightarrow S_{1/2} \Rightarrow S_{1/4}\). The following example shows that the separation axioms \(S_{1/2}\) and \(S_{1/4}\) are not in general equivalent. (They turn out to be equivalent in AT spaces, as shown in §4.)

**Example 3.2.** Let \(X\) be an infinite set and \(x_0, x_1\) be two different points of \(X\). Let \(Cof\) be the cofinite topology on \(X\), and consider the topology \(E = \{G \subseteq X : G \subseteq X \setminus \{x_0, x_1\}\} \cup \{X\}\). The topology \(\tau = Cof \cap E\) is \(S_{1/4}\) but is not \(S_{1/2}\). In fact, \([x_0]_\tau = [x_1]_\tau = \{x_0, x_1\}\) is a closed set, and for all \(y \notin \{x_0, x_1\}\), the set \([y]_\tau = \{y\}\) is kerneled but not open. Note that \(X\) is the only open set containing \(x_i\) \((i = 0, 1)\); hence, \(\tau\) is not \(T_0\).
It is worth notice that the separation axioms $S_i$ defined above are weaker than the regularity property. Recall that a topology $\tau$ is said to be regular if for each $x \in X$ and $V \in \tau(x)$, there exists $U \in \tau(x)$ such that $cl(U) \subseteq V$. Thus, $\tau$ is regular iff $\tau = \tau_0$. From our result, $\tau$ is $S_2$ iff $B(\tau) = B(\tau_0)$; it follows immediately that if $\tau$ is regular, then $\tau$ satisfies $S_2$.

To prove our main result, we need some facts concerning the classes $[x]_\tau$.

**Lemma 3.3.** The following hold for any topology $\tau$ on $X$.

(i) $[x]_\tau = cl(x) \cap ker(x)$, for all $x \in X$ (i.e., $[x]_\tau$ is $\lambda$-closed);

(ii) if $A$ is kerneled or saturated and $x \in A$, then $[x]_\tau \subseteq A$;

(iii) $cl([x]_\tau) = cl(x)$, $ker([x]_\tau) = ker(x)$, and $cl_\theta([x]_\tau) = cl_\theta(x)$, for all $x \in X$;

(iv) $[x]_\tau \subseteq cl_\theta(x) \subseteq [x]_{\tau_0}$, for all $x \in X$.

**Proof:** (i) Let $x \in X$. By definition, $y \in [x]_\tau$ iff $cl(x) = cl(y)$ iff $ker(x) = ker(y)$ iff $y \in cl(x) \cap ker(x)$.

(ii) If $A$ is kerneled (saturated, respectively) and $x \in A$, then $ker(x) \subseteq A$ ($cl(x) \subseteq A$, respectively). Thus, $[x]_\tau \subseteq A$.

(iii) Let $x \in X$. Since $x \in [x]_\tau$ and since $cl$, $ker$, and $cl_\theta$ are monotone operators, it is enough to prove that $cl([x]_\tau) \subseteq cl(x)$, $ker([x]_\tau) \subseteq ker(x)$, and $cl_\theta([x]_\tau) \subseteq cl_\theta(x)$. If $y \notin cl(x)$, there exists $V \in \tau(y)$ such that $x \notin V$. Then $x \in X \setminus V$, a closed set and thus, $[x]_\tau \subseteq X \setminus V$. Therefore, $y \notin cl([x]_\tau)$. Similarly, it can be proved that $ker([x]_\tau) = ker(x)$ for any $x \in X$. Now if $y \notin cl_\theta(x)$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $U \cap cl(V) = \emptyset$. By (ii), $[x]_\tau \subseteq U$ and thus, $[x]_\tau \cap cl(V) = \emptyset$. Then $y \notin cl_\theta([x]_\tau)$.

(iv) Let $x \in X$. The first inclusion follows from the facts that $[x]_\tau \subseteq cl_\theta([x]_\tau)$ and that $[x]_\tau$. Now given $y \in X$, if $y \notin cl_{\tau_0}(x)$, there exists a $\theta$-closed set $B$ containing $x$ such that $y \notin B$. Since $X \setminus B$ is $\theta$-open containing $y$, there is $V \in \tau(y)$ such that $cl(V) \subseteq X \setminus B$. Then $x \notin cl(V)$ and thus, $y \notin cl_\theta(x)$. Therefore, $cl_\theta(x) \subseteq cl_{\tau_0}(x)$ for all $x \in X$. On the other hand, if $y \in cl_\theta(x)$, then $x \in cl_\theta(y) \subseteq cl_{\tau_0}(y)$. Thus, $y \in ker_{\tau_0}(x)$, which proves that $cl_\theta(x) \subseteq ker_{\tau_0}(x)$ for all $x \in X$. Applying (i) to the topology $\tau_0$, one has that $[x]_{\tau_0} = cl_{\tau_0}(x) \cap ker_{\tau_0}(x)$, for all $x \in X$. Therefore, $[x]_\tau \subseteq cl_\theta(x) \subseteq [x]_{\tau_0}$, for all $x \in X$. \qed
Remark 3.4. It was proved in [1] that $\tau$ is $T_0$ iff every point is $\lambda$-closed, that is to say, $[x]_\tau = x$ for every $x \in X$. From this, it is clear that $\tau$ is $T_i$ iff it is $S_i$ and $T_0$, for $i = 1/4, 1/2, 1, or 2$.

As $B(\tau)$ is complete, then it is atomic and its atoms are given by the following result.

**Proposition 3.5.** The atoms of $B(\tau)$ are the sets $[x]_\tau$, for $x \in X$.

**Proof:** Let $C$ be the complete Boolean algebra generated by the equivalence classes $[x]_\tau$ with $x \in X$. Since the equivalence classes form a partition of $X$, then they are the atoms of $C$. Thus, it suffices to show that $B(\tau) = C$. By Lemma 3.3(i), $[x]_\tau$ is $\lambda$-closed for all $x \in X$, so $[x]_\tau \in B(\tau)$ for all $x \in X$. Thus, $C \subseteq B(\tau)$. On the other hand, if $A$ is open and $x \in A$, then $[x]_\tau \subseteq A$ (Lemma 3.3(ii)). Thus, $\tau \subseteq C$ and therefore, $B(\tau) \subseteq C$. And we are done. □

**Corollary 3.6.** $\tau$ is $T_0$ iff $B(\tau) = \mathcal{P}(X)$.

**Proof:** Since $\tau$ is $T_0$ iff $[x]_\tau = \{x\}$ for all $x \in X$ and since the singletons are the atoms of $\mathcal{P}(X)$, then the result follows from Proposition 3.5. □

**Theorem 3.7.** $B(\tau) = \overline{\tau} \lor \overline{\tau}^*$.

**Proof:** Since $\overline{\tau} \lor \overline{\tau}^* = \{A : A is an arbitrary union of kerneled and saturated sets\}$, then it is clear that $\overline{\tau} \lor \overline{\tau}^* \subseteq B(\tau)$. Now since $[x]_\tau = cl(x) \cap ker(x)$, it follows that $[x]_\tau \in \overline{\tau} \lor \overline{\tau}^*$ for all $x \in X$. Then, if $A \in B(\tau)$, $A = \bigcup_{x \in A}[x]_\tau$ (Proposition 3.5) and thus, $A \in \overline{\tau} \lor \overline{\tau}^*$ since $\overline{\tau} \lor \overline{\tau}^*$ is a topology on $X$. Therefore, $B(\tau) \subseteq \overline{\tau} \lor \overline{\tau}^*$. □

The following lemma shows other properties of the $E_\tau$-equivalence classes $[x]_\tau$, which are crucial to the remainder of this paper.

**Lemma 3.8.** Let $x \in X$.

(i) $[x]_\tau$ is closed iff $[x]_\tau$ is saturated.

(ii) $[x]_\tau$ is closed or open iff $[x]_\tau$ is $\lambda$-open.

(iii) $[x]_\tau$ is closed or kerneled iff $[x]_\tau \in CX(\tau)^*$.

(iv) $[x]_\tau$ is $\theta$-closed iff $[x]_\tau \in B(\tau_\theta)$.

**Proof:** (i) If $[x]_\tau$ is saturated, then $cl([x]_\tau) = cl(x) \subseteq [x]_\tau$, so that $[x]_\tau$ is closed. The other direction is obvious, as every closed set is saturated.
Proposition 3.9. $\tau$ satisfies $S_2$ iff $[x]_\tau$ is $\theta$-closed for all $x \in X$.

Proof: Suppose $\tau$ satisfies $S_2$. We will show that $cl_\theta([x]_\tau) = [x]_\tau$ for all $x \in X$. Let $x \in X$ and $y \in cl_\theta([x]_\tau)$. Since $cl_\theta([x]_\tau) = cl_\theta(x)$ (Lemma 3.3(iii)), it follows that $V \cap W \neq \emptyset$ for all $V \in \tau(x)$ and all $W \in \tau(y)$. It must be then that $cl(x) = cl(y)$, for otherwise, $cl(x)$ and $cl(y)$ can be separated by disjoint open sets, a contradiction. Thus, $y \in [x]_\tau$, and it follows that $cl_\theta([x]_\tau) \subseteq [x]_\tau$. The reverse inclusion is obvious by the definition of the $\theta$-closure of a set.

Conversely, suppose that $[x]_\tau$ is $\theta$-closed for all $x \in X$. Since $[x]_\tau \subseteq ker(x)$, $cl(x) \subseteq cl_\theta([x]_\tau) = cl_\theta(x) = cl_\theta([x]_\tau)$, then $[x]_\tau = ker(x) = cl(x) = cl_\theta(x)$ for all $x \in X$. Let $x, y \in X$ such that $cl(x) \neq cl(y)$. Then $y \notin [x]_\tau$, and thus, there exist $V \in \tau(x)$ and $W \in \tau(y)$ such that $V \cap W = \emptyset$. It is clear that $cl(x) = ker(x) \subseteq V$ and $cl(y) = ker(y) \subseteq W$. Therefore, $\tau$ satisfies $S_2$. □

From the above results, the separation axioms $S_i$, $i = 2, 1, 1/2$, and 1/4, can be characterized as follows.

Proposition 3.10. Let $\tau$ be a topology on $X$. Then $\tau$ satisfies

(i) $S_2$ iff $[x]_\tau \in B(\tau_0)$, for every $x \in X$;
(ii) $S_1$ iff $[x]_\tau \in \tau^*$, for every $x \in X$;
(iii) $S_{1/2}$ iff $[x]_\tau \in \lambda_\tau$, for every $x \in X$;
(iv) $S_{1/4}$ iff $[x]_\tau \in CX(\tau)^*$, for every $x \in X$. 

Proof: (i) follows immediately by Lemma 3.8(iv) and Proposition 3.9.

(ii) Recall that $\tau^*$ is the family of the saturated sets. Since $\tau$ is $S_1$ iff $[x]_\tau$ is closed for every $x \in X$, the result follows from Lemma 3.8(i).

Finally, (iii) and (iv) follow directly from the definition of the separation axioms $S_{1/2}$ and $S_{1/4}$ and Lemma 3.8(ii) and (iii), respectively.

Now we are ready to prove the main result of this section.

**Theorem 3.11.** Let $\tau$ be a topology on $X$. Then $\tau$ satisfies

(i) $S_2$ iff $B(\tau) = B(\tau_0)$;

(ii) $S_1$ iff $B(\tau) = \tau$;

(iii) $S_{1/2}$ iff $B(\tau) = \lambda^*_\tau$;

(iv) $S_{1/4}$ iff $B(\tau) = CX(\tau)$.

Proof: Let $A \subseteq B(\tau)$ such that $\emptyset \in A$ and it is closed under arbitrary unions. Since the atoms of $B(\tau)$ are the sets $[x]_\tau$, for $x \in X$, it follows immediately that $[x]_\tau \in A$ for all $x \in X$ iff $B(\tau) = A$. The result follows directly from this observation, Proposition 3.10, and the fact that $B(\tau)$ is a complete Boolean algebra.

**Corollary 3.12.** $\tau \lor \tau^*$ is $S_{1/4}$ for any topology $\tau$.

Proof: We show first that the smallest AT topology containing $\tau \lor \tau^*$ is $\tau \lor \tau^*$. In fact, since the supremum of two AT topologies is AT (see [15]), and since $\tau \lor \tau^* \supseteq \tau \lor \tau^*$, then $\tau \lor \tau^* \supseteq \tau \lor \tau^*$. The reverse inclusion follows from the fact that $\tau \lor \tau^*$ contains $\tau$ and $\tau^*$. Thus, $\tau \lor \tau^* = B(\tau)$. Now by Theorem 3.7, $\tau \lor \tau^*$ is a Boolean algebra, so $(\tau \lor \tau^*)^* = \tau \lor \tau^*$. Hence, $\tau \lor \tau^* = B(\tau \lor \tau^*)$. Note that the family of convex sets for the topology $\tau \lor \tau^*$ is precisely $\tau \lor \tau^*$. Therefore, $B(\tau \lor \tau^*) = \tau \lor \tau^* = CX(\tau \lor \tau^*)$. The result follows from Theorem 3.11(iv).

**Corollary 3.13.** $\tau$ is $S_1$ iff $\overline{\tau} = \overline{\tau^*}$ (i.e., the kerneled sets and saturated sets of $\tau$ coincide).

Proof: By Theorem 3.11, if $\tau$ is $S_1$, then $B(\tau) = \overline{\tau}$. Since $B(\tau) = \overline{\tau} \lor \tau^*$ (Theorem 3.7), then $\overline{\tau^*} \subseteq \overline{\tau}$ and thus, $\overline{\tau} = \overline{\tau^*}$. Reciprocally, if $\overline{\tau} = \overline{\tau^*}$, then $B(\tau) = \overline{\tau} \lor \tau^* = \overline{\tau}$ and so, $\tau$ is $S_1$. □
The next result is an immediate consequence of Corollary 3.6, Theorem 3.11, and the fact that a topology $\tau$ is $T_i$ iff it is $S_i$ and $T_0$, for $i = 2, 1, 1/2$, and $1/4$.

**Corollary 3.14.** Let $\tau$ be a topology on $X$. Then $\tau$ satisfies

1. $T_2$ iff $P(X) = B(\tau_0)$;
2. $T_1$ iff $P(X) = \tau$;
3. $T_{1/2}$ iff $P(X) = \lambda^*_\tau$;
4. $T_{1/4}$ iff $P(X) = CX(\tau)$.

By definitions of $\lambda$-closed sets and convex sets, it is clear that these concepts coincide on finite sets. We prove in the following lemma that these two concepts also coincide for finite unions of atoms.

**Lemma 3.15.** For every finite set $F \subset X$, $\bigcup_{x \in F}[x]_{\tau}$ is $\lambda$-closed iff it is convex.

**Proof:** Let $F = \{x_1, x_2, ..., x_n\}$, and suppose that $\bigcup_{i=1}^n[x_i]_{\tau}$ is convex. Then $\bigcup_{i=1}^n[x_i]_{\tau} = K \cap S$ for some $K \in \tau$ and $S \in \tau^*$. Since $cl(F) = \bigcup_{i=1}^n cl(x_i) \supseteq \bigcup_{i=1}^n[x_i]_{\tau}$, then $K \cap cl(F) \supseteq K \cap S$. On the other hand, since $S$ is saturated and contains $F$, it contains $cl(F)$. Thus, $K \cap S \supseteq K \cap cl(F)$. Therefore, $\bigcup_{i=1}^n[x_i]_{\tau} = K \cap cl(F)$ is a $\lambda$-closed set. The converse is obvious as every $\lambda$-closed set is convex. □

Part (ii) of the following theorem was proved differently in [1].

**Theorem 3.16.** (i) $\tau$ is $S_{1/4}$ iff $\bigcup_{x \in F}[x]_{\tau}$ is convex for every finite set $F \subset X$.

(ii) $\tau$ is $T_{1/4}$ iff every finite set is $\lambda$-closed.

**Proof:** For each $F \subseteq X$, we will denote $\bigcup_{x \in F}[x]_{\tau}$ by $[F]$.

(i) Suppose $\tau$ is $S_{1/4}$ and $F \subseteq X$ is a finite set. By Theorem 3.11(iv), $CX(\tau) = B(\tau)$. Since $[x]_{\tau} \in B(\tau)$ for each $x \in X$, and since $B(\tau)$ is a Boolean algebra, then $[F] \in B(\tau)$ and thus, $[F]$ is convex. Conversely, suppose that $[F]$ is convex for every finite set $F$. To show that $\tau$ is $S_{1/4}$, it is enough to show that $B(\tau) \subseteq CX(\tau)$. If $A \notin CX(\tau)$, there exist $x, z \in A$ and $y \in X \setminus A$ such that $x \preceq y$ and $y \preceq z$. Since the set $[x]_{\tau} \cup [z]_{\tau}$ is convex and contains $x$ and $z$, then it contains $y$. It follows that $A \notin B(\tau)$.

(ii) follows from (i) and Lemma 3.15. □
The following proposition is a folklore fact. We give a proof using results from this section.

**Proposition 3.17.** \( \tau \) is \( T_2 \) iff \( \tau_\theta \) is \( T_1 \) iff \( \tau_\theta \) is \( T_{1/2} \) iff \( \tau_\theta \) is \( T_0 \) [5].

*Proof:* By Corollary 3.11, \( \tau \) is \( T_2 \) iff \( P(X) = \mathcal{B}(\tau_\theta) \), and by Corollary 3.6, this equality holds iff \( \tau_\theta \) is \( T_0 \). On the other hand, if \( \tau \) is \( T_2 \), then \( \tau \) is \( T_0 \) and \( S_2 \). Thus, \( \{x\} = [x]_\tau \) is \( \theta \)-closed for all \( x \in X \), and therefore, \( \tau_\theta \) is \( T_1 \). The rest of the equivalences are obvious. \( \square \)

4. Additional results

As an application of the previous section, we include some results about \( \lambda \)-spaces, AT spaces, and the order theoretical property.

4.1. \( \lambda \)-spaces

In [1], \( \lambda \)-spaces were defined as those spaces \( (X, \tau) \) for which the family \( \lambda_\tau \) of the \( \lambda \)-open sets is a topology. It is easy to prove that \( (X, \tau) \) is a \( \lambda \)-space iff \( V \cap S \in \lambda_\tau \) for all \( V \in \tau \) and \( S \in \tau^* \). Since \( \tau \) is \( S_{1/2} \) iff \( \mathcal{B}(\tau) = \lambda_\tau \), every \( S_{1/2} \) space is a \( \lambda \)-space. Moreover, every \( \lambda \)-space is \( S_{1/4} \) as shown in next result.

**Proposition 4.1.** Every \( \lambda \)-space is \( S_{1/4} \). Hence, every \( T_0 \), \( \lambda \)-space is \( T_{1/4} \).

*Proof:* If \( (X, \tau) \) is a \( \lambda \)-space, then the finite union of \( \lambda \)-closed sets is a \( \lambda \)-closed set. Thus, for every finite set \( F \subset X \), the set \( \bigcup_{x \in F} [x]_\tau \) is \( \lambda \)-closed, therefore convex. By Theorem 3.16(i), \( \tau \) is \( S_{1/4} \). \( \square \)

Let \( LC(\tau) \) be the family of \( \tau \)-locally closed sets in \( X \), i.e., \( LC(\tau) = \{V \cap F, \text{ with } V \in \tau \text{ and } F \in \tau^*\} \). The following theorem gives a characterization of \( \lambda \)-spaces.

**Theorem 4.2.** The following are equivalent.

(i) \( (X, \tau) \) is a \( \lambda \)-space.
(ii) \( LC(\tau) \subseteq \lambda_\tau \).
(iii) \( \lambda_\tau = \tau \lor \tau^* \).
(iv) \( cl_\lambda \) is a Kuratowski closure operator.

*Proof:* Suppose \( (X, \tau) \) is a \( \lambda \)-space. Then \( \lambda_\tau \) is a topology on \( X \) and thus, the intersection of two \( \lambda \)-open sets is a \( \lambda \)-open set. Since
$\tau \cup \tau^* \subseteq \lambda_\tau$, then $A \in LC(\tau)$ implies $A$ is $\lambda$-open. Therefore, (i) $\Rightarrow$ (ii).

Recall that $A \in \lambda_\tau$ implies that $A = V \cup S$ for some $V \in \tau$ and $S \in \tau^*$; thus, it is obvious that $\lambda_\tau \subseteq \tau \vee \tau^*$. On the other hand, since any set of the form $V \cap S$, with $V \in \tau$ and $S \in \tau^*$, can be written as an arbitrary union of locally closed sets, (namely $V \cap S = \bigcup_{x \in S} V \cap cl(x)$), it follows that $LC(\tau)$ is a basis for the topology $\tau \vee \tau^*$. Thus, $LC(\tau) \subseteq \lambda_\tau$ implies $\tau \vee \tau^* \subseteq \lambda_\tau$, since the family $\lambda_\tau$ is closed under arbitrary unions. Therefore, (ii) $\Rightarrow$ (iii).

Now suppose that $\lambda_\tau = \tau \vee \tau^*$. Then $\lambda_\tau$ is a topology. To prove that $cl_\lambda$ is a Kuratowski operator, one has to prove that it is additive on $X$. Let $A, B \subseteq X$. Since $cl_\lambda(A) \cup cl_\lambda(B)$ is a $\lambda$-closed set containing $A \cup B$, then $cl_\lambda(A \cup B) \subseteq cl_\lambda(A) \cup cl_\lambda(B)$. The reverse inclusion follows from the monotonicity of the $cl_\lambda$ operator. Therefore, (iii)$\Rightarrow$ (iv). The equivalence of (i) and (iv) is trivial. □

The property of being a $\lambda$-space is strictly placed between $S_{1/2}$ and $S_{1/4}$, as the following examples show.

**Example 4.3 ($S_{1/4} \not\Rightarrow \lambda$-space).** Consider the $S_{1/4}$ space $(X, \tau)$ given in Example 3.2. Let $A$ be an infinite proper subset of $X \setminus \{x_0, x_1\}$ and let $B = A \cup \{x_0, x_1\}$. Note that $A = \cap \{X \setminus \{y, x_0, x_1\} : y \notin B\}$ is kerneled and so $\lambda$-closed. Also, the set $\{x_0, x_1\}$ is $\lambda$-closed since it is closed. But $B$ is not $\lambda$-closed, since $cl(B) = X = ker(B)$, and hence, $cl_\lambda(B) = X \neq B$. Therefore, $(X, \tau)$ is not a $\lambda$-space.

**Example 4.4 ($\lambda$-space $\not\Rightarrow S_{1/2}$).** Let $\rho$ be the topology of the digital line on the set of integers numbers $Z$ (i.e., the topology generated by the sets $\{2n-1, 2n, 2n+1\}_{n \in Z}$). This is an AT and $T_{1/2}$ topology. Let $\tau$ be the topology on $Z$ generated by $Z \setminus cl_\rho(F)$ for $F \subseteq Z$ finite. Note that $\overline{\tau} = \rho$. In fact, since $\tau \subseteq \rho$ and $\rho$ is an AT topology, then $\overline{\tau} \subseteq \rho$. On the other hand, since any basic $\rho$-open set $\{2n-1, 2n, 2n+1\}$ can be written as $\bigcap_{k \neq n} Z \setminus \{2k\}$, a $\tau$-kerneled set, then $\rho \subseteq \tau$. The topology $\tau$ is $T_0$ but not $T_{1/2}$, as the open points of $(Z, \rho)$ are neither $\tau$-open nor $\tau$-closed. Thus, $\tau$ is not $S_{1/2}$. It is straightforward to check that $LC(\tau) \subseteq \tau^*$. Since closed sets are $\lambda$-open, then $(X, \tau)$ is a $\lambda$-space by Theorem 4.2.
4.2 AT SPACES

If \( \tau \) is an AT topology (i.e., \( \tau = \tau^* \)), then the smallest complete Boolean algebra \( \mathbb{B}(\tau) \) containing \( \tau \) is just the topology \( \tau \lor \tau^* \). We prove below that some of the low separation axioms defined in section 3 are indistinguishable for AT topologies. For instance, \( S_1/2 \) and \( S_1/4 \) are equivalent, and \( S_1 \) and \( S_2 \) are equivalent to the regularity property. Note that, for AT topologies, \( \text{CO}(\tau) = \text{CO}(\tau\theta) = \tau\theta = \mathbb{B}(\tau\theta) \). This shows, in particular, that \( \tau\theta \) is a complete Boolean algebra.

**Proposition 4.5.** Let \( \tau \) be an AT topology. Then,

(i) \( \tau \) is \( S_2 \) iff \( \tau \) is \( S_1 \) iff \( \tau \) is regular;

(ii) \( \tau \) is \( S_{1/2} \) iff \( \tau \) is \( S_{1/4} \) iff \( (X, \tau) \) is a \( \lambda \)-space.

**Proof:** (i) If \( \tau \) is \( S_1 \), then \( \mathbb{B}(\tau) = \tau \) (Theorem 3.11) and so \( \mathbb{B}(\tau) = \text{CO}(\tau) \). Therefore, \( \tau \) is regular. Since regular implies \( \lambda \)-property, the conclusion follows.

(ii) Since, for AT topologies, saturated sets are closed, then \( \lambda^* = CX(\tau) \). Thus, the equivalence \( S_{1/2} \) iff \( \tau \) is \( S_{1/4} \) is a consequence of Theorem 3.11. The other equivalence follows from the fact that \( \lambda \)-space property is placed between \( S_{1/2} \) and \( S_{1/4} \). \( \square \)

The separation axioms \( S_{1/2} \) and \( S_1 \) are not equivalent in AT spaces. In fact, the topology \( \rho \) of the digital line (see Example 4.4) is an AT and \( T_{1/2} \) topology, thus \( S_{1/2} \) and \( T_0 \). But, being not discrete, \( \rho \) is not \( T_1 \). Therefore, it is not \( S_1 \).

We end this section with a result about a notion taken from [16]. A property \( P \) (like a separation axiom) is said to be order theoretical if the following holds: Given topologies \( \tau \) and \( \rho \) such that \( \leq_{\tau} = \leq_{\rho} \), if \( (X, \tau) \) satisfies \( P \), then \( (X, \rho) \) also satisfies \( P \). Since \( cl_{\tau}(x) = cl_{\tau}(x) \) for any \( x \in X \), then \( \leq_{\tau} = \leq_{\rho} \) iff \( \tau = \rho \) for any pair of topologies \( \tau \) and \( \rho \) on \( X \). Thus, a property \( P \) is order theoretical if the following holds: \( (X, \tau) \) satisfies \( P \) iff \( (X, \tau) \) satisfies \( P \).

**Theorem 4.6.** The separation axioms \( T_0 \), \( T_1 \), \( T_{1/4} \), \( S_1 \), and \( S_{1/4} \) are order theoretical.

**Proof:** The result follows immediately from Corollary 3.6, Theorem 3.11, Corollary 3.14, and the facts that \( \mathbb{B}(\tau) = \mathbb{B}(\tau) \) and \( CX(\tau) = CX(\tau) \). \( \square \)
Remark 4.7. The separation axioms $S_2$, $S_{1/2}$, and $\lambda$-space are not order theoretical.

Since $B(\tau_\theta) \subseteq CO(\tau) = (\tau)_\theta = B((\tau)_\theta) \subseteq B(\tau)$, then if $\tau$ is $S_2$, it follows that $\overline{\tau}$ is $S_2$. Now let $\tau$ be the cofinite topology on an infinite set. The topology $\tau$ is not $S_2$, but $\tau$ is the discrete topology which is obviously $S_2$ (note that any singleton $\{x\}$ can be written as $\bigcap_{y \neq x} X \setminus \{y\}$, a kerneled set). Therefore, $S_2$ is not order theoretical.

Note that, for AT topologies, the family of $\lambda$-closed sets coincides with the family of convex sets; thus, $\lambda^*_\tau = CX(\overline{\tau}) = CX(\tau)$. If $\tau$ is $S_{1/2}$, then $\lambda^*_\tau = B(\tau)$, and so $\lambda^*_\tau = B(\tau) = B(\overline{\tau})$ which shows that $\overline{\tau}$ is $S_{1/2}$. But the fact that $\overline{\tau}$ is $S_{1/2}$ does not imply that $\tau$ is $S_{1/2}$. Consider, for instance, the digital topology $\rho$ on $\mathbb{Z}$, and let $\tau$ be the non $S_{1/2}$ topology given in Example 4.4. Since $\tau = \rho$ and $\rho$ is $T_{1/2}$, then $\tau$ is, in particular, $S_{1/2}$. Therefore, $S_{1/2}$ is not order theoretical.

If $(X, \tau)$ is a $\lambda$-space, then $\tau$ is $S_{1/4}$ by Proposition 4.5. The above theorem implies that $\overline{\tau}$ is $S_{1/4}$. Since $\overline{\tau}$ is an AT topology, then $(X, \overline{\tau})$ is $\lambda$-space by Theorem 4.2(ii). But Example 4.3 shows that $\lambda$-space and $S_{1/4}$ are not equivalent. Then the fact that $(X, \overline{\tau})$ is a $\lambda$-space does not imply that $(X, \tau)$ is a $\lambda$-space. Hence, the property of being a $\lambda$-space is not order theoretical.

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References


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