Remarks on the Generic Existence of Ultrafilters on \( \omega \)

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Electronically published on February 12, 2009
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OF ULTRAFILTERS ON $\omega$

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Abstract. The purpose of this note is to contrast the generic existence of certain kinds of ultrafilters on $\omega$ with the existence of $2^\omega$-many of them. First, we prove that it is consistent with ZFC that there are $2^\omega$-many $Q$-points but $Q$-points do not exist generically. This answers in the negative a question by R. Michael Canjar. Then we define the strong generic existence of a class of ultrafilters and show that the strong generic existence of selective ultrafilters is equivalent to their generic existence. However, we prove a result that implies that for several classes of ultrafilters, including $P$-points and nowhere dense ultrafilters, the strong generic existence of $P$-points is not equivalent to their generic existence.

1. Preliminaries

We use standard set theoretic notation. We say that $\mathcal{A} \subseteq [\omega]^\omega$ has the strong finite intersection property (SFIP) provided that the intersection of any finite subfamily is infinite. The filter generated by $\mathcal{A}$ is denoted $\langle \mathcal{A} \rangle$. The letter $\mathcal{F}$ will always denote a filter on $\omega$ containing the cofinite filter. A basis for $\mathcal{F}$ is a family $\mathcal{B} \subseteq \mathcal{F}$ such that for every $F \in \mathcal{F}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq F$. Given any filter $\mathcal{F}$ let $\chi(\mathcal{F})$ be the minimum cardinality of a basis of $\mathcal{F}$. This $\chi(\mathcal{F})$ is called the character of $\mathcal{F}$. We say that $\mathcal{F}$

2000 Mathematics Subject Classification. Primary 03E05; Secondary 03E65, 04A20, 54A25.

Key words and phrases. dominating number and covering number for the meager ideal, generic existence, $P$-points, $Q$-points, selective ultrafilters.
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is $\kappa$-generated provided that $\chi(\mathcal{F}) = \kappa$ and $\mathcal{F}$ is $< \kappa$-generated provided that $\chi(\mathcal{F}) < \kappa$. The letters $\mathcal{U}$ and $\mathcal{V}$ will always denote nonprincipal ultrafilters on $\omega$. The cardinal $u$ is defined as

$$u = \min\{|B| : B \text{ is basis of an ultrafilter on } \omega\}.$$

Two filters $\mathcal{F}_0$ and $\mathcal{F}_1$ are orthogonal provided that there exists an $X \in [\omega]^\omega$ such that $X \in \mathcal{F}_0$ and $\omega \setminus X \in \mathcal{F}_1$. This is denoted $\mathcal{F}_0 \perp \mathcal{F}_1$. An ultrafilter $\mathcal{U}$ is a $Q$-point provided that for every finite-to-one $f: \omega \to \omega$ there exists a $U \in \mathcal{U}$ such that $f|U$ is one-to-one. On the other hand, $\mathcal{U}$ is rapid provided that for every $f: \omega \to \omega$ there is a $U \in \mathcal{U}$ such that $|U \cap f(n)| \leq n$ for every $n < \omega$. Every $Q$-point is rapid but not every rapid ultrafilter is a $Q$-point. An ultrafilter $\mathcal{U}$ is a $P$-point provided that for every partition $\mathcal{P}$ of $\omega$ either $\mathcal{P} \cap U \neq \emptyset$ or there exists a $U \in \mathcal{U}$ such that $|U \cap P| < \omega$ for every $P \in \mathcal{P}$. We will call such a $U \in \mathcal{U}$ a partial pseudo-selector of $\mathcal{P}$. If in this definition we require instead that there exists a $U \in \mathcal{U}$ such that $|U \cap P| \leq 1$ for every $P \in \mathcal{P}$, we obtain the definition of a selective or Ramsey ultrafilter, and we will call such a $U \in \mathcal{U}$ a partial selector of $\mathcal{P}$. It is well known that an ultrafilter is selective if and only if it is both a $P$-point and a $Q$-point. An ultrafilter which is both a $P$-point and a rapid ultrafilter is called semiselective. If $f, g \in \omega^\omega$, we declare $f \preceq g$ when $|\{n < \omega : f(n) > g(n)\}| < \omega$. A family $\mathcal{G} \subseteq \omega^\omega$ is dominating provided that for every $f \in \omega^\omega$ there is a $g \in \mathcal{G}$ such that $f \preceq g$, and it is unbounded provided that there is no single $f \in \omega^\omega$ such that $g \preceq f$ for every $g \in \mathcal{G}$. The cardinals $\mathfrak{d}$, $\mathfrak{b}$, $\text{cov}(\mathcal{M})$, and $\text{non}(\mathcal{N})$ denote the minimum cardinality of a dominating family, an unbounded family, a family of meager sets whose union covers $\mathbb{R}$, and a non-measure zero subset of $\mathbb{R}$, respectively. These cardinals are related as shown in Figure 1, where $\kappa \to \lambda$ means $\kappa \leq \lambda$.

![Figure 1. A fragment of Cichon's diagram.](image-url)
2. Generic and strong generic existence of ultrafilters

Definition 2.1. Let $\mathcal{C}$ be a class of ultrafilters on $\omega$ and let $\kappa$ be an uncountable cardinal. We abbreviate by $GE(\mathcal{C}, \kappa)$ the statement “every $< \kappa$-generated filter can be extended to an ultrafilter in $\mathcal{C}$.” Here, $GE$ stands for “generic existence.”

In what follows, we will denote by $P$, $Q$, $R$, $S$, and $SS$ the classes of $P$-points, $Q$-points, rapid, selective, and semi-selective ultrafilters, respectively.

The next three propositions characterize the generic existence of these ultrafilters in terms of $\text{cov}(\mathcal{M})$ and $d$.

Proposition 2.2 (Ketonen [13]).

$$GE(P, c) \iff d = c.$$

Proposition 2.3 (Canjar [12]).

$$GE(Q, d) \iff \text{cov}(\mathcal{M}) = d \iff GE(R, d).$$

Proposition 2.4 (Canjar [12]; Bartoszynski and Judah [10]).

$$GE(S, c) \iff \text{cov}(\mathcal{M}) = c \iff GE(SS, c).$$

In [12, p. 240], R. Michael Canjar asked, Assuming that $c$ is regular, does the existence of $2^c$-many selective ultrafilters imply $GE(S, c)$? We answered this negatively in [17] by constructing a model of ZFC where $c = \omega_2$, and there are $2^c$-many selective ultrafilters but $\text{cov}(\mathcal{M}) < c$. The same question for $c$ singular is an unpublished result by James E. Baumgartner who noticed that in the Bell-Kunen model described in [6], $c = \omega_{\omega_1}$, $\text{cov}(\mathcal{M}) = \omega_1$, and there are $2^c$-many selective ultrafilters on $\omega$.

Definition 2.5. Let $M$ be a model of ZFC. A forcing notion $\mathbb{P}$ is $\omega^\omega$-bounding provided that for every $\mathbb{P}$-generic filter $G$ over $M$ and for every $f \in \omega^\omega \cap M[G]$, there exists a $g \in \omega^\omega \cap M$ such that $\forall n < \omega f(n) < g(n)$.

Proposition 2.6 (Millán [17]). There is a model $N$ of ZFC such that

$$N \models "c = \omega_2 + |S| = 2^c + \neg GE(S, c)."$$
Proof: Let $M$ be such that $\models \text{ZFC} + \text{CH} + 2^{\omega_1} = 2^{\omega_2} = \omega_3$.

If $P \in M$ is the partial order to add $\omega_2$-many Sacks reals iteratively with countable supports and $G$ is $P$-generic over $M$, then

$M[G] \models \text{ZFC} + \mathfrak{c} = \omega_2 + 2^{\omega_1} = 2^{\omega_2} = \omega_3$.

Now, CH in $M$ implies that $\mathfrak{d}^M = \omega_1^M$. Since $P$ is $\omega_\omega$-bounding and proper, we have that $\omega_1^M[G] = \omega_1^M = \mathfrak{d}^M = \mathfrak{d}^M[G]$. In particular, $M[G] \models \text{cov}(M) < \mathfrak{c}$.

By Proposition 2.4, $M[G] \models \neg GE(S, \mathfrak{c})$.

To see that $M[G] \models |S| = 2^\mathfrak{c}$, use CH to construct in $M$, $2^{\omega_1}$-many selective ultrafilters. Then invoke a theorem by Baumgartner and Richard Laver [4, Theorem 4.5] to extend these to $2^{\omega_1}$-many selective ultrafilters in $M[G]$. Therefore,

$M[G] \models \mathfrak{c} \text{ is regular} + |S| = 2^\mathfrak{c} + \neg GE(S, \mathfrak{c})$.

Hence, $N = M[G]$ works. □

Canjar also asked [12], Does the existence of $2^\mathfrak{d}$-many $Q$-points imply $GE(Q, \mathfrak{d})$?

We will answer this question negatively by constructing a model of \text{ZFC}+|Q| = 2^\mathfrak{c} + \neg GE(Q, \mathfrak{d})$.

**Definition 2.7.** Let $f : \omega \to \omega$ be any function. We say that $U \subseteq \omega$ is $f$-rare if $f(m) < n$ for every $m, n \in U$ with $m < n$.

**Definition 2.8.** A family $U \subseteq [\omega]^{\omega}$ is rare if for every $f : \omega \to \omega$ there exists a $U_f \in U$ which is $f$-rare.

**Proposition 2.9** (Mathias [16]; Taylor, unpublished. See also Blass [8]). An ultrafilter $U$ on $\omega$ is a $Q$-point if and only if $U$ is a rare family.

Let $\mathcal{U}$ and $\mathcal{V}$ be two families of subsets of $\omega$ and let $\Psi(\mathcal{U}, \mathcal{V})$ be an abbreviation of the statement, “$\mathcal{U} \neq \mathcal{V}$ and both $\mathcal{U}$ and $\mathcal{V}$ are $Q$-points.”

**Lemma 2.10.** Let $M$ be a transitive model of ZFC, let $\mathcal{U}, \mathcal{V} \in M$ such that $M \models \Psi(\mathcal{U}, \mathcal{V})$, and let $\mathbb{P} \in M$ be an $\omega^\omega$-bounding forcing notion. Then, for any $\mathbb{P}$-generic filter $G$ over $M$,

$M[G] \models \text{ZFC} + \exists \mathcal{U} \exists \mathcal{V} \ (\mathcal{U} \subseteq \mathcal{U} \land \mathcal{V} \subseteq \mathcal{V} \land \Psi(\mathcal{U}, \mathcal{V}))$.

Proof: Let $\mathcal{U}$ be a $Q$-point in $M$, let $G$ be a $\mathbb{P}$-generic filter over $M$, and let $f \in \omega^\omega \cap M[G]$. Then there exists a $g \in \omega^\omega \cap M$ such that $f(n) < g(n)$ for every $n < \omega$. Since $M \models \Psi(\mathcal{U}, \mathcal{V})$,
we can find a \( U_g \in \mathcal{U} \) which is \( g \)-rare. Since \( f \) is dominated by \( g \), we conclude that \( U_g \) is also \( f \)-rare. Therefore, \( M[G] \models \text{“} \mathcal{U} \text{ is rare.} \) If \( M \models \text{“} \mathcal{U} \neq V \text{,”} \) then there exists a \( U \in (\mathcal{U} \setminus \mathcal{V}) \cap M \subseteq (\mathcal{U} \setminus \mathcal{V}) \cap M[G]. \) This implies that \( U \in \mathcal{U} \) and \( \omega \setminus U \in \mathcal{V}. \) If \( \mathcal{U} \) and \( \mathcal{V} \) are ultrafilters in \( M[G], \) extending \( \mathcal{U} \) and \( \mathcal{V}, \) respectively, then \( \mathcal{U} \) and \( \mathcal{V} \) are distinct \( Q \)-points. Hence, \( M[G] \models \text{“} \Psi(\mathcal{U}, \mathcal{V}). \text{”} \)

\[ \square \]

**Corollary 2.11.** If an \( \omega^\omega \)-bounding forcing notion preserves \( P \)-points, then it preserves selective ultrafilters.

**Theorem 2.12.** There are models \( N_i \) of ZFC for \( i = 0, 1 \) such that

(a) \( N_0 \models \text{“} \text{ZFC} + \varepsilon = \omega_2 + |Q| = 2^\varepsilon + \neg \text{GE}(Q, \emptyset) \text{,”} \) and

(b) \( N_1 \models \text{“} \text{ZFC} + \varepsilon = \omega_{\omega_1} + |Q| = 2^\varepsilon + \neg \text{GE}(Q, \emptyset) \text{.”} \)

\[ \text{Proof:} \] Suppose that \( \kappa \in \{ \omega_2, \omega_{\omega_1} \}. \) If \( M \models \text{“} \text{ZFC} + \text{GCH} \text{,”} \) let \( \mathbb{P} = \mathbb{P}(\kappa) \in M \) be the notion of forcing for adding \( \kappa \)-many Cohen reals and let \( G \) be a \( \mathbb{P} \)-generic filter over \( M; \) then we have that \( M[G] \models \text{“} \kappa = \varepsilon = \text{cov}(\mathcal{M}) = 0 \text{.”} \) By Proposition 3.2, it follows that \( M[G] \models \text{“} \varepsilon = \kappa + |Q| = 2^\varepsilon \text{.”} \) Let \( Q = \mathbb{B}(\kappa) \in M[G] \) be the measure algebra for adding \( \kappa \)-many random reals. Let \( H \) be a \( Q \)-generic filter over \( M[G]. \) Then \( \mathbb{Q} \) is \( \omega^\omega \)-bounding and \( M[G][H] \models \text{“} \varepsilon = \kappa = 0 \text{.”} \) By Lemma 2.10, we can extend each of the \( Q \)-points existing in \( M[G] \) to at least one \( Q \)-point in \( M[G][H] \) obtaining \( 2^\varepsilon \)-many \( Q \)-points altogether. Hence, \( M[G][H] \models \text{“} |Q| = 2^\varepsilon \text{.”} \) Let \( S \in M[G][H] \) be the set formed by the first \( \omega_1 \)-many random reals added. Since \( S \) is a Sierpinski set in \( M[G][H], \) it is non-measurable, so \( M[G][H] \models \text{“} \text{non}(\mathcal{N}) = \omega_1 \text{.”} \) On the other hand, \( \text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{N}); \) hence, \( M[G][H] \models \text{“} \text{cov}(\mathcal{M}) < 0 \text{.”} \) By Proposition 2.3, \( M[G][H] \models \text{“} |Q| = 2^\varepsilon \wedge \neg \text{GE}(Q, \emptyset) \text{.”} \) Therefore, models \( N_0 = M[G][H], \) when \( \kappa = \omega_2, \) and \( N_1 = M[G][H], \) when \( \kappa = \omega_{\omega_1}, \) satisfy the conclusion of the theorem. \[ \square \]

### 3. Generic versus Strong Generic Existence of Ultrafilters on \( \omega \)

In this section we show that for some classes of ultrafilters \( \mathcal{C}, \) \( \text{GE}(\mathcal{C}, \varepsilon) \) fails to be a good indicator of the abundance of ultrafilters from \( \mathcal{C}. \) As an alternative, we propose \( \text{SGE}(\mathcal{C}, \varepsilon) \) instead.

**Definition 3.1.** \( \text{SGE}_\lambda(\mathcal{C}, \kappa) \) abbreviates the statement, “every \( \kappa \)-generated filter can be extended to \( 2^\lambda \)-many ultrafilters in \( \mathcal{C}, \)” where \( \text{SGE} \) stands for “strong generic existence.” When \( \lambda = \varepsilon, \) we
drop the subindex and write \( SGE(\mathcal{C}, \kappa) \). We will use \( SGE(\mathcal{C}, \kappa) \) to abbreviate “the strong generic existence of ultrafilters in \( \mathcal{C} \).”

**Proposition 3.2** (Millán [17]).

\[
SGE(Q, \mathfrak{d}) \iff \operatorname{cov}(\mathcal{M}) = \mathfrak{d}.
\]

Actually, we proved in [17] that the identity \( \operatorname{cov}(\mathcal{M}) = \mathfrak{d} \) implies that every \( < \mathfrak{d} \)-generated filter can be extended to \( 2^\kappa \)-many \( \kappa \)-generated \( Q \)-points. The other direction follows from Proposition 2.3 and Lemma 3.5(b) below.

As an immediate consequence of Proposition 3.2, we have the following dichotomy result.

**Corollary 3.3.** Suppose that \( \kappa \leq \operatorname{cov}(\mathcal{M})^+ \), then either there is a \( \kappa \)-generated \( Q \)-point or there are no \( Q \)-points at all.

**Proof:** If \( \kappa \leq \operatorname{cov}(\mathcal{M})^+ \), then \( \operatorname{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \kappa \leq \operatorname{cov}(\mathcal{M})^+ \). Suppose that \( \operatorname{cov}(\mathcal{M}) = \mathfrak{d} \). Then we are done by Proposition 3.2 and the remark below it. If \( \operatorname{cov}(\mathcal{M}) < \mathfrak{d} \), then \( \mathfrak{d} = \kappa \). Since, by Proposition 2.9, every \( Q \)-point has character \( \geq \mathfrak{d} \) and either there is a \( Q \)-point (in which case it has character \( \kappa \)) or there are no \( Q \)-points at all, this completes the argument. \( \square \)

Notice that propositions 2.3 and 3.2 can be combined to obtain the following.

**Proposition 3.4.** \( SGE(Q, \mathfrak{d}) \iff GE(Q, \mathfrak{d}) \).

**Lemma 3.5.** Let \( \mathcal{C} \) be a class of ultrafilters and let \( \kappa, \lambda, \) and \( \mu \) be cardinals. Then

(a) \( SGE_0(\mathcal{C}, \kappa) \iff GE(\mathcal{C}, \kappa) \),

(b) \( \lambda \leq \mu \Rightarrow SGE\mu(\mathcal{C}, \kappa) \Rightarrow SGE\lambda(\mathcal{C}, \kappa) \), and

(c) \( SGE_1(\mathcal{C}, \kappa) \iff (GE(\mathcal{C}, \kappa) \land \kappa \leq \mathfrak{u}) \).

**Proof:** Parts (a) and (b) are obvious. For part (c), one implication follows from parts (a) and (b) and the fact that \( \mathfrak{u} < \kappa \Rightarrow \neg SGE_1(\mathcal{C}, \kappa) \). For the other implication, let \( \mathcal{F} \) be a filter with \( \chi(\mathcal{F}) < \kappa \). Since \( \kappa \leq \mathfrak{u} \), \( \mathcal{F} \) cannot be an ultrafilter and there exists an \( X \in [\omega]^\omega \) such that \( \mathcal{F} \cup \{X\} \) and \( \mathcal{F} \cup \{\omega \setminus X\} \) both have the SFIP. Let \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) be the filters generated by \( \mathcal{F} \cup \{X\} \) and \( \mathcal{F} \cup \{\omega \setminus X\} \), respectively. Then \( \chi(\mathcal{F}_0) = \chi(\mathcal{F}_1) = \chi(\mathcal{F}) < \kappa \). So we can use \( GE(\mathcal{C}, \kappa) \) to extend \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) to ultrafilters \( \mathcal{U}_0 \) and \( \mathcal{U}_1 \) in \( \mathcal{C} \). By our choice of \( X \), these ultrafilters are distinct. \( \square \)
Definition 3.6. We call a class $C$ of ultrafilters on $\omega$ to be $\kappa$-inductive provided that there exist formulas $\langle \phi_\xi(Y) : \xi < \kappa \rangle$ such that for every $U$

$$U \in C \iff \forall \xi < \kappa \exists U \in \mathcal{U} \phi_\xi(U).$$

Lemma 3.7. The classes $P$, $S$, and $Q$ are $\mathfrak{c}$-inductive.

Proof: To see that $P$ is $\mathfrak{c}$-inductive, let $\langle P_\xi : \xi < \mathfrak{c} \rangle$ be a listing of the partitions of $\omega$ into infinitely many pieces and consider for every $\xi < \mathfrak{c}$ the formula

$$\phi_\xi(Y) \iff [(\exists G \in [P_\xi]^{<\omega})(Y \subseteq \bigcup G) \lor (\forall P \in P_\xi)(|Y \cap P| < \omega)].$$

For $S$, replace $|Y \cap P| < \omega$ by $|Y \cap P| \leq 1$ in the formulas above. For $Q$, let $\langle P_\xi : \xi < \mathfrak{c} \rangle$ be a listing of the partitions of $\omega$ into finite pieces and consider $\phi_\xi(Y) \iff (\forall P \in P_\xi)(|Y \cap P| \leq 1)$ for every $\xi < \mathfrak{c}$. □

Lemma 3.8. The class $Q$ is $\mathfrak{d}$-inductive.

Proof: Let $\langle f_\xi \in \omega^\omega : \xi < \mathfrak{d} \rangle$ be a dominating family and consider for each $\xi < \mathfrak{d}$ the formula

$$\phi_\xi(Y) \iff (\forall m < \omega)(\forall n < \omega) (m, n \in Y \land m < n \Rightarrow f_\xi(m) < n).$$

Then the $\mathfrak{d}$-inductivity of $Q$ follows from Proposition 2.9. □

Theorem 3.9. If $\kappa \geq 1$ and $C$ is a $\kappa$-inductive class of ultrafilters, then

$$SGE_\kappa(C, \kappa) \iff (GE(C, \kappa) \land \kappa \leq \mathfrak{u}).$$

Proof: It is obvious that $SGE_\kappa(C, \kappa)$ implies both $GE(C, \kappa)$ and $\kappa \leq \mathfrak{u}$. So suppose that $G(C, \kappa)$ and $\kappa \leq \mathfrak{u}$ hold and that $\mathcal{F}$ is a $\kappa$-generated filter. We will construct inductively a tree $\langle \mathcal{F}_s : s \in 2^{<\kappa} \rangle$ of filters satisfying the following requirements for every $\xi < \kappa$ and $s \in 2^\xi$.

1. $\mathcal{F}_0 = \mathcal{F}$;
2. $\mathcal{F}_{s\gamma} \subseteq \mathcal{F}_s$ for every $\gamma < \xi$;
3. $\chi(\mathcal{F}_s) \leq \max\{\chi(\mathcal{F}), |\xi|\}$;
4. $\mathcal{F}_{s^{\langle 0 \rangle}} \perp \mathcal{F}_{s^{\langle 1 \rangle}}$;
5. $\mathcal{F}_s = \bigcup\{\mathcal{F}_{s\gamma} : \gamma < \xi\}$ if $\xi$ is limit; and
6. there exists $X_\xi^i \in \mathcal{F}_{s^{\langle i \rangle}}$ such that $\phi_\xi(X_\xi^i)$ hold for $i = 0, 1$. 

□
If this construction can be completed, then for every $g \in 2^\kappa$, let $\mathcal{U}_g$ be an ultrafilter extending the filter $\mathcal{F}_g = \bigcup\{\mathcal{F}_g|\xi : \xi < \kappa\}$. By conditions (1), (4), and (6), the ultrafilters $\mathcal{U}_g$ extend $\mathcal{F}$, are pairwise distinct, and are all in $\mathcal{C}$. To see that this construction can be completed, we need only to check the inductive hypothesis for the successor ordinal case. Suppose that $s \in 2^\xi$ and that $\mathcal{F}_s$ has been defined. We want to define $\mathcal{F}_s \wedge \langle 0 \rangle$ and $\mathcal{F}_s \wedge \langle 1 \rangle$. By the induction hypothesis, $\chi(\mathcal{F}_s) < \kappa \leq u$, so we can find a $Y \in [\omega]^{\omega}$ such that $\mathcal{F}_s \cup \{Y\}$ and $\mathcal{F}_s \cup \{\omega \setminus Y\}$ have both SFIP. Since $\mathcal{F}^*_0 = \langle \mathcal{F}_s \cup \{Y\} \rangle$ and $\mathcal{F}^*_1 = \langle \mathcal{F}_s \cup \{\omega \setminus Y\} \rangle$, we have that $\chi(\mathcal{F}^*_0) = \chi(\mathcal{F}^*_1) = \chi(\mathcal{F}_s) < \kappa$. Also, since $GE(\mathcal{C}, \kappa)$ holds, there exist $U_i \in \mathcal{C}$ extending $\mathcal{F}^*_i$ for $i = 0, 1$. Thus, it is possible to pick $X^i_\xi \in U_i$ such that $\phi(\theta^i_\xi)$ for $i = 0, 1$. Put $\mathcal{F}_s \wedge \langle 0 \rangle = \langle \mathcal{F}_s \cup \{Y, X^0_\xi\} \rangle$ and $\mathcal{F}_s \wedge \langle 1 \rangle = \langle \mathcal{F}_s \cup \{\omega \setminus Y, X^1_\xi\} \rangle$. Then these filters satisfy the requirements. \hfill $\square$

**Corollary 3.10.** If $\mathcal{C}$ is $\mathfrak{c}$-inductive like $P$, $Q$, or $S$, then

$$SGE(\mathcal{C}, \mathfrak{c}) \iff SGE_1(\mathcal{C}, \mathfrak{c}).$$

*Proof:* This follows from Lemma 3.5(c), Lemma 3.7, and Theorem 3.9. \hfill $\square$

**Corollary 3.11.** $SGE(S, \mathfrak{c}) \iff GE(S, \mathfrak{c})$.

*Proof:* This follows from Theorem 3.9 and Proposition 2.4. \hfill $\square$

**Corollary 3.12.** $SGE(P, \mathfrak{c}) \iff \min\{\mathfrak{u}, \mathfrak{d}\} = \mathfrak{c}$.

*Proof:* This follows from Theorem 3.9 and Proposition 2.2. \hfill $\square$

By a theorem of Jason Aubrey [1], $\min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$. Here, $\mathfrak{r}$ is the refinement or reaping number. (See [7].) Therefore, we can rephrase Corollary 3.12 as

**Corollary 3.13.** $SGE(P, \mathfrak{c}) \iff \min\{\mathfrak{r}, \mathfrak{d}\} = \mathfrak{c}$.

**Theorem 3.14.** There is a model $N$ of ZFC such that

$$N \models "GE(P, \mathfrak{c}) \land \neg SGE(P, \mathfrak{c})."$$

*Proof:* Let $M$ be such that $M \models "ZFC + GCH,\"$ and consider in $M$ a countable support forcing iteration $\langle (\mathbb{P}_\alpha, \dot{Q}_\alpha) : \alpha < \omega_2 \rangle$ such that

$$\forall \alpha < \omega_2 \quad \forces_{\alpha} "\dot{Q}_\alpha \simeq \text{rational perfect set forcing}"$$
THE GENERIC EXISTENCE OF ULTRAFILTERS ON \( \omega \) (see [19]; [10, p. 360]; and [9]), and let \( G \) be a \( \mathbb{P}_{\omega_2} \)-generic filter over \( M \). Then \( P \)-points in \( M \) generate \( P \)-points in \( M[G] \) and
\[
M[G] \models \text{"Every } P \text{-point is } \omega_1 \text{-generated } + d = c = \omega_2 = 2^{\omega_1}."
\]
(See [9].) Therefore,
\[
M[G] \models \text{"} GE(P, c) \land |P| = c. \text{"
}\]
Hence, model \( N = M[G] \) works.

\[\square\]

**Corollary 3.15.** \( \text{Con}(ZFC) \Rightarrow \text{Con}(ZFC + GE(P, c) + \neg SGE(P, c)) \).

4. **Other classes of ultrafilters**

Theorem 3.9 can be applied to get a similar characterization for \( SGE(\mathcal{C}, c) \) as in Corollary 3.13 for other classes of ultrafilters as well. These depend, of course, on the characterization of \( GE(\mathcal{C}, c) \) in terms of cardinal invariants.

**Definition 4.1** (Baumgartner [3]). Let \( X \) be a non-empty set and let \( I \subseteq \mathcal{P}(X) \) be a set containing the singletons and closed under subsets. An ultrafilter \( U \) on \( \omega \) is an \( I \)-ultrafilter provided that for every \( f : \omega \to X \) there exists a \( U \in U \) such that \( f[U] \in I \).

If \( X = 2^\omega \) and \( I \) is the ideal of countable closed, nowhere-dense, measure-zero subsets of \( 2^\omega \), then \( I \)-ultrafilters are called **countable closed, nowhere-dense, and measure-zero ultrafilters**, respectively. If \( X = \omega_1 \) and \( I \) is the ideal of \( \sigma \)-compact subsets of \( \omega_1 \), then the \( I \)-ultrafilters are called **\( \sigma \)-compact ultrafilters**. If \( \alpha < \omega_1 \), put \( I_\alpha = \{ A \subseteq \omega_1 : o.t(A) \leq \alpha \} \) and \( J_\alpha = \{ A \subseteq \omega_1 : o.t(A) < \alpha \} \). If \( I = I_\alpha \) or \( I = J_\alpha \) for some \( \alpha < \omega_1 \), then \( I \)-ultrafilters are called **ordinal ultrafilters**. Let \( O, CC, ND, MZ, \) and \( K_\sigma \) denote the classes of ordinal, countable closed, nowhere-dense, measure-zero, and \( \sigma \)-compact ultrafilters, respectively.

**Lemma 4.2.** If \( \mathcal{C} \in \{ O, CC, ND, MZ, K_\sigma \} \), then \( \mathcal{C} \) is \( c \)-inductive.

**Proof:** Let \( \langle f_\xi : \xi < c \rangle \) be a listing of \( X^\omega \) and consider the family of formulas \( \langle \phi_\xi(Y) : \xi < c \rangle \) where \( \phi_\xi(Y) \iff f_\xi[Y] \in I \).

We refer the reader to [5] and [11] for proofs of the following propositions.

**Proposition 4.3** (Brendle [11]). If \( \mathcal{C} \in \{ O, CC \} \), then
\[
GE(\mathcal{C}, c) \iff d = c.
\]
Proposition 4.4 (Brendle [11]). \(GE(ND, c) \iff \text{cof}(\mathcal{M}) = c\).

Proposition 4.5 (Brendle [11]). \(GE(MZ, c) \iff \text{cof}(\mathcal{E}, \mathcal{M}) = c\).

Proposition 4.6 (Barney [5]). \(GE(K_\sigma, c) \iff d = c\).

Theorem 4.7. If \(\mathcal{C} \in \{O, CC, K_\sigma\}\), then
\[
\begin{align*}
(a) & \quad SGE(\mathcal{C}, c) \iff \min\{u, d\} = c; \\
(b) & \quad SGE(ND, c) \iff \min\{u, \text{cof}(\mathcal{M})\} = c; \\
(c) & \quad SGE(MZ, c) \iff \min\{u, \text{cof}(\mathcal{E}, \mathcal{M})\} = c.
\end{align*}
\]

Proof: Apply Theorem 3.9 and propositions 4.1 and 4.4 for (a), Proposition 4.2 for (b), and Proposition 4.3 for (c). \(\Box\)

Theorem 4.8. There is a model \(N\) of \(ZFC\) such that if \(\mathcal{C} \in \{O, CC, ND, MZ, K_\sigma\}\), then
\[
N \models \text{"}GE(\mathcal{C}, c) \land \neg SGE(\mathcal{C}, c)\text{"}.
\]

Proof: The model from Theorem 3.14 works since it is known that in this model, \(d = \text{cof}(\mathcal{M}) = \text{cof}(\mathcal{E}, \mathcal{M}) = c\). \(\Box\)

References


