A Concrete Co-Existential Map
That Is Not Confluent

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Electronically published on June 23, 2009
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Abstract. We give a concrete example of a co-existential map between continua that is not confluent.

Introduction

In [1], Paul Bankston gives an example of a co-existential map that is not confluent. The construction is rather involved and does not produce a concrete example of such a map. A lot of effort is needed to get the main ingredient, to wit, a co-diagonal map that is not monotone.

The purpose of this note is to show that one can write down a concrete map between two rather simple continua that is co-existential and not confluent. It will be clear from the construction that the range space admits co-diagonal maps that are not confluent and, a fortiori, not monotone.

1. Preliminaries

In the interest of brevity, we try to keep the notation down to the bare minimum.
2.1 Ultra-copowers and associated maps

Given a compact space $Y$ and a set $I$, we consider the Čech-Stone compactification $\beta(Y \times I)$, where $I$ carries the discrete topology. There are two useful maps associated with $\beta(Y \times I)$: the Čech-Stone extensions of the projections $\pi_Y : Y \times I \to Y$ and $\pi_I : Y \times I \to I$. Given an ultrafilter $u$ on $I$, we write $Y_u = \beta\pi_I^{-1}(u)$ and we let $q_u = \beta\pi_Y | Y_u$. In the terminology of [1], the space $Y_u$ is the ultra-copower of $Y$ by the ultrafilter $u$ and $q_u : Y_u \to Y$ is the associated co-diagonal map. A map $f : X \to Y$ between compact spaces is co-existential if there are a set $I$, an ultrafilter $u$ on $I$, and a map $g : Y_u \to X$ such that $q_u = f \circ g$.

These notions can be seen as dualizations of notions from model theory and they offer inroads to the study of compact Hausdorff spaces by algebraic and, in particular, lattice-theoretic means.

2.2 Two notions from continuum theory

On a first-order algebraic level there is not much difference between $Y$ and $Y_u$: they have elementarily equivalent lattice-bases for their closed sets; the map $A \mapsto Y_u \cap \text{cl}_\beta(A \times I)$ is an elementary embedding of such bases. It is, therefore, not unreasonable to expect that the co-diagonal map $q_u$ be well-behaved. For example, one could expect it to be confluent, which means that if $C$ is a subcontinuum of $Y$ then every component of $q_u^{-1}[C]$ would be mapped onto $C$ by $q_u$. Certainly some component of $q_u^{-1}[C]$ is mapped onto $C$: the component that contains $Y_u \cap \text{cl}_\beta(C \times I)$ (this shows that $q_u$ is weakly confluent). Intuitively, there should be no difference between the components, so all should be mapped onto $C$. The example below disproves this intuition.

In [1], Bankston gives (references for) other reasons why it is of interest to know whether co-diagonal and co-existential maps are confluent.

2. The example

We start with the closed infinite broom [3, 120, p. 139]

$$B = ([0, 1] \times \{0\}) \cup \bigcup_{n \in \omega} H_n$$

where $H_n = \{\langle t, t/2^n \rangle : 0 \leq t \leq 1\}$ is the $n$th hair of the broom.
The range space is $B$ with the limit hair extended to have length 2:
$$Y = B \cup ([1, 2] \times \{0\}).$$

We denote the extended hair $[0, 2] \times \{0\}$ by $H_\omega$.

The domain of the map is $B$ with an extra hair of length 2 along the $y$-axis:
$$X = B \cup \{(0) \times [0, 2]\}.$$

The map $f : X \to Y$ is the (more-or-less) obvious one:
$$f(x, y) = \begin{cases} 
(x, y) & x \in B \\
(y, 0) & x = 0.
\end{cases}$$

Thus, $f$ is the identity on $B$ and it rotates the points on the extra hair over $-\frac{1}{2}\pi$.

**Claim 1.** The map $f$ is not confluent.

**Proof:** This is easy. The components of the preimage of the continuum $C = [1, 2] \times \{0\}$ are the interval $\{0\} \times [1, 2]$ and the singleton $\{(1, 0)\}$; the latter does not map onto $C$. □

**Claim 2.** The map $f$ is co-existential.

**Proof:** We need to find an ultrafilter $u$ and a map $g : Y_u \to X$ such that $f \circ g$ is the co-diagonal map $q_u : Y_u \to Y$. In fact, any free ultrafilter $u$ on $\omega$ will do.

We define two closed subsets $F$ and $G$ of $Y \times \omega$ and define $g$ on the intersections $F_u = Y_u \cap \text{cl}_B F$ and $G_u = Y_u \cap \text{cl}_B G$ separately. We set
$$F = \bigcup_{n \in \omega} \left( \bigcup_{k \leq n} (H_k \times \{n\}) \right)$$
and
$$G = \bigcup_{n \in \omega} \left( \bigcup_{n < k \leq \omega} (H_k \times \{n\}) \right).$$

Note that $F \cup G = Y \times \omega$ and that $F \cap G = \{(0, 0)\} \times \omega$, so that $F_u \cup G_u = Y_u$ and $F_u \cap G_u$ consists of one point, the (only) accumulation point of $F \cap G$ in $Y_u$.

It is an elementary verification that $q_u[F_u] = B$ and $q_u[G_u] = H_\omega$.

This allows us to define $g : Y_u \to X$ by cases: on $F_u$, we define $g$ to be just $q_u$, and on $G_u$, we define $g = R \circ q_u$, where $R$ rotates the plane over $\frac{1}{2}\pi$. These definitions agree at the point in $F_u \cap G_u$ and...
give continuous maps on $F_u$ and $G_u$, respectively. Therefore, the combined map $g : Y_u \to X$ is continuous as well.

This also shows that the co-diagonal map $q_u$ is not confluent; no component of the preimage under $g$ of $\langle 1, 0 \rangle$ is mapped onto $C$.

**Remark.** In [2], Bankston shows that if a continuum $K$ is such that every co-existential map onto $K$ is confluent, then every $K$ must be connected im kleinen at each of its cut points. The continuum $Y$ above is connected im kleinen at all cut points but one: the point $\langle 1, 0 \rangle$. So $Y$ does not qualify as a counterexample to the converse.

To obtain a counterexample, multiply $X$ and $Y$ by the unit interval and multiply $f$ by the identity. The proof that the new map is co-existential but not confluent is an easy adaptation of the proof that $f$ has these properties. Since $Y$ has no cut points, it is connected im kleinen at all of them.

**References**

