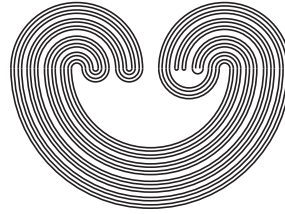


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## DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

by

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## DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

JACK S. CALCUT AND JOHN D. McCARTHY

**ABSTRACT.** For a locally path connected topological space, the topological fundamental group is discrete if and only if the space is semilocally simply-connected. While functoriality of the topological fundamental group, with target the category of topological groups, remains an open question in general, the topological fundamental group is always a homogeneous space.

### 1. INTRODUCTION

The concept of a natural topology for the fundamental group appears to have originated with Witold Hurewicz [8] in 1935. It received further attention in 1950 by James Dugundji [2] and more recently by Daniel K. Biss [1], Paul Fabel [3], [4], [5], [6], and others. The purpose of this note is to prove the following folklore theorem.

**Theorem 1.1.** *Let  $X$  be a locally path connected topological space. The topological fundamental group  $\pi_1^{\text{top}}(X)$  is discrete if and only if  $X$  is semilocally simply-connected.*

Theorem 5.1 of [1] is Theorem 1.1 without the hypothesis of local path connectedness. However, a counterexample of Fabel [6] shows that this stronger result is false. Fabel [6] also proves a weaker

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version of Theorem 1.1, assuming that  $X$  is locally path connected and a metric space. In this note we remove the metric hypothesis.

Our proof proceeds from first topological principles, making no use of rigid covering fibrations [1] nor even of classical covering spaces. We make no use of the functoriality of the topological fundamental group, a property which was also a main result in [1, Corollary 3.4] but, in fact, is unproven [5, pp. 188–189]. Beware that the misstep in the proof of Proposition 3.1 in [1], namely the assumption that the product of quotient maps is a quotient map, is repeated in Theorem 2.1 of [7].

In general, the homeomorphism type of the topological fundamental group depends on a choice of basepoint. We say that  $\pi_1^{\text{top}}(X)$  is *discrete*, without reference to a basepoint, provided  $\pi_1^{\text{top}}(X, x)$  is discrete for each  $x \in X$ . If  $x$  and  $y$  are connected by a path in  $X$ , then  $\pi_1^{\text{top}}(X, x)$  and  $\pi_1^{\text{top}}(X, y)$  are homeomorphic. This fact was proved in Proposition 3.2 of [1], and a detailed proof is provided for completeness in section 4 of this paper. Theorem 1.1 now immediately implies the following.

**Corollary.** *Let  $X$  be a path connected and locally path connected topological space. The topological fundamental group  $\pi_1^{\text{top}}(X, x)$  is discrete for some  $x \in X$  if and only if  $X$  is semilocally simply-connected.*

As mentioned above, it is open whether  $\pi_1^{\text{top}}$  is a functor from the category of pointed topological spaces to the category of topological groups. The unsettled question is whether multiplication

$$\begin{array}{ccc} \pi_1^{\text{top}}(X, x) \times \pi_1^{\text{top}}(X, x) & \xrightarrow{\mu} & \pi_1^{\text{top}}(X, x) \\ ([f], [g]) & \longmapsto & [f] \cdot [g] \end{array}$$

is continuous. By Theorem 1.1, if  $X$  is locally path connected and semilocally simply-connected, then  $\pi_1^{\text{top}}(X, x)$ , and, hence, the product  $\pi_1^{\text{top}}(X, x) \times \pi_1^{\text{top}}(X, x)$  are discrete and so  $\mu$  is trivially continuous. Continuity of  $\mu$ , in general, remains an interesting question.

Lemma 5.1 below shows that if  $(X, x)$  is an arbitrary pointed topological space, then left and right multiplication by any fixed element in  $\pi_1^{\text{top}}(X, x)$  are continuous self maps of  $\pi_1^{\text{top}}(X, x)$ . Therefore,  $\pi_1^{\text{top}}(X, x)$  acts on itself by left and right translation as a group of self homeomorphisms. Clearly, these actions are transitive. Thus, we obtain the following result.

**Theorem 1.2.** *Let  $(X, x)$  be a pointed topological space. Then  $\pi_1^{\text{top}}(X, x)$  is a homogeneous space.*

This note is organized as follows. Section 2 contains definitions and conventions, section 3 proves two lemmas and Theorem 1.1, section 4 addresses change of basepoint, and section 5 shows left and right translation are homeomorphisms.

## 2. DEFINITIONS AND CONVENTIONS

By convention, neighborhoods are open. Unless stated otherwise, homomorphisms are inclusion induced.

Let  $X$  be a topological space and  $x \in X$ . A neighborhood  $U$  of  $x$  is *relatively inessential* (in  $X$ ) provided  $\pi_1(U, x) \rightarrow \pi_1(X, x)$  is trivial.  $X$  is *semilocally simply-connected* at  $x$  provided there exists a relatively inessential neighborhood  $U$  of  $x$ .  $X$  is *semilocally simply-connected* provided it is so at each  $x \in X$ . A neighborhood  $U$  of  $x$  is *strongly relatively inessential* (in  $X$ ) provided  $\pi_1(U, y) \rightarrow \pi_1(X, y)$  is trivial for every  $y \in U$ .

The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Consequently, if  $A$  and  $B$  are any subsets of  $X$  such that  $x \in A \subset B \subset X$  and  $\pi_1(B, x) \rightarrow \pi_1(X, x)$  is trivial, then  $\pi_1(A, x) \rightarrow \pi_1(X, x)$  is trivial as well. This observation justifies the convention that neighborhoods are open.

If  $X$  is locally path connected and semilocally simply-connected, then each  $x \in X$  has a path connected relatively inessential neighborhood  $U$ . Such a  $U$  is necessarily a strongly relatively inessential neighborhood of  $x$ , as the reader may verify (see for instance, [9, Exercise 5, p. 330]).

Let  $(X, x)$  be a pointed topological space and let  $I = [0, 1] \subset \mathbb{R}$ . The space

$$C_x(X) = \{f : (I, \partial I) \rightarrow (X, x) \mid f \text{ is continuous}\}$$

is endowed with the compact-open topology. The function

$$\begin{array}{ccc} C_x(X) & \xrightarrow{q} & \pi_1(X, x) \\ f & \longmapsto & [f] \end{array}$$

is surjective, so  $\pi_1(X, x)$  inherits the quotient topology, and one writes  $\pi_1^{\text{top}}(X, x)$  for the resulting *topological fundamental group*. Let  $e_x \in C_x(X)$  denote the constant map. If  $f \in C_x(X)$ , then  $f^{-1}$  denotes the path defined by  $f^{-1}(t) = f(1 - t)$ .

### 3. PROOF OF THEOREM 1.1

We prove two lemmas and then Theorem 1.1.

**Lemma 3.1.** *Let  $(X, x)$  be a pointed topological space. If  $\{[e_x]\}$  is open in  $\pi_1^{\text{top}}(X, x)$ , then  $x$  has a relatively inessential neighborhood in  $X$ .*

*Proof:* The quotient map  $q$  is continuous and  $\{[e_x]\} \subset \pi_1^{\text{top}}(X, x)$  is open, so  $q^{-1}([e_x]) = [e_x]$  is open in  $C_x(X)$ . Therefore,  $e_x$  has a basic open neighborhood

$$(3.1) \quad e_x \in V = \bigcap_{n=1}^N V(K_n, U_n) \subset [e_x] \subset C_x(X),$$

where each  $K_n \subset I$  is compact, each  $U_n \subset X$  is open, and each  $V(K_n, U_n)$  is a subbasic open set for the compact-open topology on  $C_x(X)$ . We will show that

$$U = \bigcap_{n=1}^N U_n$$

is a relatively inessential neighborhood of  $x$  in  $X$ . Clearly,  $U$  is open in  $X$  and, by (3.1),  $x \in U$ . Finally, let  $f : (I, \partial I) \rightarrow (U, x)$ . For each  $1 \leq n \leq N$ , we have

$$f(K_n) \subset U \subset U_n.$$

Thus,  $f \in [e_x]$  by (3.1), so  $[f] = [e_x]$  is trivial in  $\pi_1(X, x)$ .  $\square$

**Lemma 3.2.** *Let  $(X, x)$  be a pointed topological space and let  $f \in C_x(X)$ . If  $X$  is locally path connected and semilocally simply-connected, then  $\{[f]\}$  is open in  $\pi_1^{\text{top}}(X, x)$ .*

*Proof:* As  $q$  is a quotient map, we must show that  $q^{-1}([f]) = [f]$  is open in  $C_x(X)$ . So let  $g \in [f]$ . For each  $t \in I$ , let  $U_t$  be a path connected relatively inessential neighborhood of  $g(t)$  in  $X$ . The sets  $g^{-1}(U_t)$ , where  $t \in I$ , form an open cover of  $I$ . Let  $\lambda > 0$  be a Lebesgue number for this cover. Choose  $N \in \mathbb{N}$  so that  $1/N < \lambda$ . For each  $1 \leq n \leq N$ , let

$$I_n = \left[ \frac{n-1}{N}, \frac{n}{N} \right] \subset I.$$

Reindex the  $U_t$ 's so that

$$g(I_n) \subset U_n \text{ for each } 1 \leq n \leq N.$$

The  $U_n$ 's are not necessarily distinct, nor does the proof require this condition. For each  $1 \leq n \leq N$ , let  $W_n$  denote the path component of  $U_n \cap U_{n+1}$  containing  $g(n/N)$ , so

$$(3.2) \quad g\left(\frac{n}{N}\right) \in W_n \subset (U_n \cap U_{n+1}) \subset X.$$

Consider the basic open set

$$(3.3) \quad V = \left( \bigcap_{n=1}^N V(I_n, U_n) \right) \cap \left( \bigcap_{n=1}^{N-1} V\left(\left\{\frac{n}{N}\right\}, W_n\right) \right) \subset C_x(X).$$

By construction,  $g \in V$ . It remains to show that  $V \subset [f]$ . So, let  $h \in V$ . As  $[g] = [f]$ , it suffices to show that  $[h] = [g]$ .

By (3.3) we have

$$(3.4) \quad \begin{aligned} h(I_n) &\subset U_n \quad \text{for each } 1 \leq n \leq N \text{ and} \\ h\left(\frac{n}{N}\right) &\in W_n \quad \text{for each } 1 \leq n \leq N-1. \end{aligned}$$

For each  $1 \leq n \leq N-1$ , let  $\gamma_n : I \rightarrow W_n$  be a continuous path such that

$$\begin{aligned} \gamma_n(0) &= h\left(\frac{n}{N}\right) \quad \text{and} \\ \gamma_n(1) &= g\left(\frac{n}{N}\right), \end{aligned}$$

which exists by (3.2) and (3.4). Let  $\gamma_0 = e_x$  and  $\gamma_N = e_x$ . For each  $1 \leq n \leq N$ , define

$$\begin{aligned} I &\xrightarrow{s_n} I_n \\ t &\longmapsto \frac{1}{N}t + \frac{n-1}{N} \end{aligned}$$

and let

$$\begin{aligned} g_n &= g \circ s_n \quad \text{and} \\ h_n &= h \circ s_n. \end{aligned}$$

So,  $g_n$  and  $h_n$  are affine reparameterizations of  $g|_{I_n}$  and  $h|_{I_n}$ , respectively. For each  $1 \leq n \leq N$ ,

$$\delta_n = g_n * \gamma_n^{-1} * h_n^{-1} * \gamma_{n-1}$$

is a loop in  $U_n$  based at  $g_n(0)$  (see Figure 1). As  $U_n$  is a strongly rel-

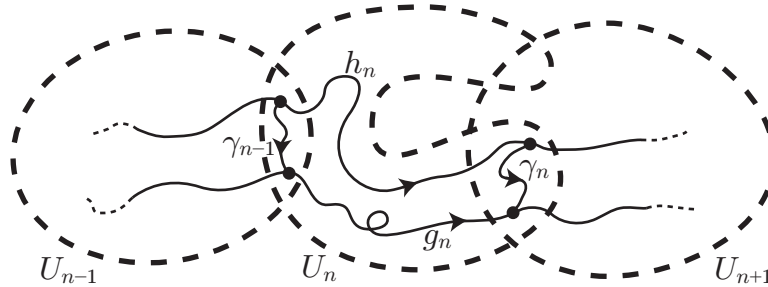


FIGURE 1. Loop  $\delta_n = g_n * \gamma_n^{-1} * h_n^{-1} * \gamma_{n-1}$  in  $U_n$  based at  $g_n(0)$ .

atively inessential neighborhood,  $[\delta_n] = 1 \in \pi_1(X, g_n(0))$ . Therefore,  $g_n$  and  $\gamma_{n-1}^{-1} * h_n * \gamma_n$  are path homotopic. In  $\pi_1(X, x)$ , we have

$$\begin{aligned} [h] &= [h_1 * h_2 * \cdots * h_N] \\ &= [\gamma_0^{-1} * h_1 * \gamma_1 * \gamma_1^{-1} * h_2 * \gamma_2 * \cdots * \gamma_{N-1}^{-1} * h_N * \gamma_N] \\ &= [g_1 * g_2 * \cdots * g_N] \\ &= [g], \end{aligned}$$

proving the lemma. □

In the previous proof, the second collection of subbasic open sets in (3.3) is essential. Figure 2 shows two loops  $g$  and  $h$  based

at  $x$  in the annulus  $X = S^1 \times I$ . All conditions in the proof are satisfied, except  $g(1/N)$  and  $h(1/N)$  fail to lie in the same connected component of  $U_1 \cap U_2$ . Clearly,  $g$  and  $h$  are not homotopic loops.

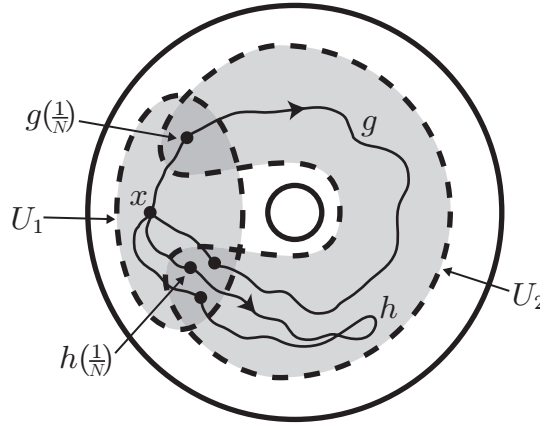


FIGURE 2. Loops  $g$  and  $h$  based at  $x$  in the annulus  $X$ .

*Proof of Theorem 1.1:* First, assume  $\pi_1^{\text{top}}(X)$  is discrete and let  $x \in X$ . By definition,  $\pi_1^{\text{top}}(X, x)$  is discrete, so  $\{[e_x]\}$  is open in  $\pi_1^{\text{top}}(X, x)$ . By Lemma 3.1,  $x$  has a relatively inessential neighborhood in  $X$ . The choice of  $x \in X$  was arbitrary, so  $X$  is semilocally simply-connected.

Next, assume  $X$  is semilocally simply-connected and let  $x \in X$ . Points in  $\pi_1^{\text{top}}(X, x)$  are open by Lemma 3.2, so  $\pi_1^{\text{top}}(X, x)$  is discrete. The choice of  $x \in X$  was arbitrary, so  $\pi_1^{\text{top}}(X)$  is discrete.  $\square$

#### 4. BASEPOINT CHANGE

**Lemma 4.1.** *Let  $X$  be a topological space and  $x, y \in X$ . If  $x$  and  $y$  lie in the same path component of  $X$ , then  $\pi_1^{\text{top}}(X, x)$  and  $\pi_1^{\text{top}}(X, y)$  are homeomorphic.*

*Proof:* Let  $\gamma : I \rightarrow X$  be a continuous path with  $\gamma(0) = y$  and  $\gamma(1) = x$ . Define the function

$$\begin{aligned} C_y(X) &\xrightarrow{\Gamma} C_x(X) \\ f &\longmapsto (\gamma^{-1} * f) * \gamma. \end{aligned}$$



First, we show that  $\Gamma$  is continuous. Let  $I_1 = [0, 1/4]$ ,  $I_2 = [1/4, 1/2]$ , and  $I_3 = [1/2, 1]$ . Define the affine homeomorphisms

$$\begin{array}{ccc} I_1 \xrightarrow{s_1} I & I_2 \xrightarrow{s_2} I & I_3 \xrightarrow{s_3} I \\ t \longmapsto 4t & t \longmapsto 4t - 1 & t \longmapsto 2t - 1 \end{array}$$

and note that

$$\begin{array}{ll} I \xrightarrow{\Gamma(f)} X & \\ t \longmapsto \gamma^{-1} \circ s_1(t) & 0 \leq t \leq \frac{1}{4} \\ t \longmapsto f \circ s_2(t) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ t \longmapsto \gamma \circ s_3(t) & \frac{1}{2} \leq t \leq 1. \end{array}$$

Consider an arbitrary subbasic open set

$$V = V(K, U) \subset C_x(X).$$

Observe that  $\Gamma(f) \in V$  if and only if

$$(4.1) \quad \gamma^{-1} \circ s_1(K \cap I_1) \subset U,$$

$$(4.2) \quad f \circ s_2(K \cap I_2) \subset U, \text{ and}$$

$$(4.3) \quad \gamma \circ s_3(K \cap I_3) \subset U.$$

Define the subbasic open set

$$V' = V(s_2(K \cap I_2), U) \subset C_y(X).$$

Observe that  $f \in V'$  if and only if (4.2) holds. As conditions (4.1) and (4.3) are independent of  $f$ , either  $\Gamma^{-1}(V) = \emptyset$  or  $\Gamma^{-1}(V) = V'$ . Thus,  $\Gamma$  is continuous. Next, consider the diagram

$$\begin{array}{ccc} C_y(X) & \xrightarrow{\Gamma} & C_x(X) \\ q_y \downarrow & & \downarrow q_x \\ \pi_1^{\text{top}}(X, y) & \xrightarrow{\pi(\Gamma)} & \pi_1^{\text{top}}(X, x). \end{array}$$

The composition  $q_x \circ \Gamma$  is constant on each fiber of  $q_y$ , so there is a unique set function making the diagram commute, namely  $\pi(\Gamma) : [f] \mapsto [\Gamma(f)]$ . As  $q_y$  is a quotient map, the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that  $\pi(\Gamma)$  is continuous. It is well known that  $\pi(\Gamma)$  is a bijection [9, Theorem 2.1, p. 327]. Repeating the above argument with the roles of

$x$  and  $y$  interchanged and the roles of  $\gamma$  and  $\gamma^{-1}$  interchanged, we see that  $\pi(\Gamma)^{-1}$  is continuous. Thus,  $\pi(\Gamma)$  is a homeomorphism as desired.  $\square$

5. TRANSLATION

**Lemma 5.1.** *Let  $(X, x)$  be a pointed topological space. If  $[f] \in \pi_1^{\text{top}}(X, x)$ , then left and right translation by  $[f]$  are self homeomorphisms of  $\pi_1^{\text{top}}(X, x)$ .*

*Proof:* Fix  $[f] \in \pi_1^{\text{top}}(X, x)$  and consider left translation by  $[f]$  on  $\pi_1^{\text{top}}(X, x)$

$$\begin{array}{ccc} \pi_1^{\text{top}}(X, x) & \xrightarrow{L_{[f]}} & \pi_1^{\text{top}}(X, x) \\ [g] & \longmapsto & [f] \cdot [g]. \end{array}$$

Plainly,  $L_{[f]}$  is a bijection of sets. Consider the commutative diagram

$$(5.1) \quad \begin{array}{ccc} C_x(X) & \xrightarrow{L_f} & C_x(X) \\ q \downarrow & & \downarrow q \\ \pi_1^{\text{top}}(X, x) & \xrightarrow{L_{[f]}} & \pi_1^{\text{top}}(X, x), \end{array}$$

where  $L_f$  is defined by

$$\begin{array}{ccc} C_x(X) & \xrightarrow{L_f} & C_x(X) \\ g & \longmapsto & f * g. \end{array}$$

First, we show  $L_f$  is continuous. Let  $I_1 = [0, 1/2]$  and  $I_2 = [1/2, 1]$ . Define the affine homeomorphisms

$$\begin{array}{ccc} I_1 & \xrightarrow{s_1} & I \\ t & \longmapsto & 2t \end{array} \quad \begin{array}{ccc} I_2 & \xrightarrow{s_2} & I \\ t & \longmapsto & 2t - 1 \end{array}$$

and note that

$$\begin{array}{ccc} I & \xrightarrow{f * g} & X \\ t & \longmapsto & f \circ s_1(t) \quad 0 \leq t \leq \frac{1}{2} \\ t & \longmapsto & g \circ s_2(t) \quad \frac{1}{2} \leq t \leq 1. \end{array}$$

Consider an arbitrary subbasic open set

$$V = V(K, U) \subset C_x(X).$$

Observe that  $f * g \in V$  if and only if

$$(5.2) \quad f \circ s_1(K \cap I_1) \subset U \text{ and}$$

$$(5.3) \quad g \circ s_2(K \cap I_2) \subset U.$$

Define the subbasic open set

$$V' = V(s_2(K \cap I_2), U) \subset C_x(X).$$

Observe that  $g \in V'$  if and only if (5.3) holds. As condition (5.2) is independent of  $g$ , either  $L_f^{-1}(V) = \emptyset$  or  $L_f^{-1}(V) = V'$ . Thus,  $L_f$  is continuous. The composition  $q \circ L_f$  is constant on each fiber of the quotient map  $q$  and (5.1) commutes, so the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that  $L_{[f]}$  is continuous.

Applying the previous argument to  $f^{-1}$ , we get  $L_{[f]}^{-1} = L_{[f^{-1}]}$  is continuous and  $L_{[f]}$  is a homeomorphism. The proof for right translation is almost identical.  $\square$

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