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by

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ABSTRACT. We show that the removal of one point from 2^{ω_1} gives a counterexample to a conjecture of Ishiu on D-spaces. We also show that Martin's Axiom implies that there are no Lindelöf non-D-spaces that can be written as union of less than continuum many compact subspaces. Finally we show that the property of being a D-space is preserved by forcing with trees of height ω.

An *open neighborhood assignment* (*ONA*) on a topological space X is a function N which assigns to each point $x \in X$ an open set $N(x)$ containing x. Given an ONA N on a space X, and subset Y of X, we let $N[Y]$ denote $\bigcup\{N(x) \mid x \in Y\}$. A space X is a *D-space* [11] if for every ONA N on X there is a closed discrete $C \subseteq X$ such that $N[C] = X$. These spaces were introduced by van Dowen in 1979 [11], and while they have attracted a lot of attention in recent years [2, 3, 4, 6, 7, 8, 9, 10, 13, 14], many basic questions remain open [12]. Probably the best known open question is whether every regular Lindelöf space is a D-space (see [16]).

In the first section we prove that removing one point from 2^{ω_1} gives a counterexample to a conjecture of Tetsuya Ishiu, as the resulting space is irreducible but not a D-space. In the second section, we prove that, assuming Martin's Axiom, there are no

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"small" Lindelöf non-D-spaces, where "small" means a union of less than continuum many compact subspaces. Finally, in the third section we consider the effects of forcing with trees of height ω. For instance, we show that if T is such a tree and X is a D-space, then X remains a D-space after forcing with T.

We assume throughout that all toplogical spaces are T_1.

1. Irreducibility and the Revised Range Conjecture

Tetsuya Ishiu proposed what he called the *Revised Range Conjecture*, asserting that every topological space X has a basis $\mathcal{B}$ such that for any two ONA's N_0, N_1 (on X) with the same range $R \subseteq \mathcal{B}$, there is a closed discrete set C_0 such that $N_0[C_0] = X$ if and only if there is a closed discrete set C_1 such that $N_1[C_1] = X$. We will see in this section that this conjecture is false.

A topological space X is said to be *irreducible* [1] if for every open cover $\mathcal{O}$ of X there is an open cover $\mathcal{O}'$ such that each element of $\mathcal{O}'$ is contained in a member of $\mathcal{O}$ and contains a point not in any other member of $\mathcal{O}'$ (such an $\mathcal{O}'$ is said to be a *minimal open refinement* of $\mathcal{O}$).

Lemma 1.1. *Let X be an irreducible space in which every nonempty open set has the same cardinality. If the Revised Range Conjecture holds for X, then X is a D-space.*

Proof. Let N be an ONA on X. We may assume that the range of N is contained in a basis $\mathcal{B}$ witnessing the Revised Range Conjecture for X. Since X is irreducible, there exists a minimal open refinement $\mathcal{O}'$ of the range of N covering X. For each $O \in \mathcal{O}'$ pick a point in O not in any other member of $\mathcal{O}'$, and let Y be the set of picked points. Then Y is a closed discrete set, and we can define a partial ONA N' on Y by letting $N'(y)$ be any member of the range of N containing the member of $\mathcal{O}'$ containing y, for each $y \in Y$. It suffices now to extend N' to an ONA on all of X with the same range as N. Since Y is closed discrete, each open set has intersection of size $|X|$ with the complement of Y. The range of N has cardinality $\kappa \leq |X|$. Let $\langle B_\alpha : \alpha < \kappa \rangle$ be a wellordering of the range of N, and choose points $\langle x_\alpha : \alpha < \kappa \rangle$ such that each $x_\alpha \in B_\alpha \setminus (Y \cup \{x_\beta : \beta < \alpha\})$, and define $N'(x_\alpha) = B_\alpha$ for each $\alpha < \kappa$. For each $x \in X \setminus (Y \cup \{x_\alpha : \alpha < \kappa\})$, let $N'(x) = N(x)$. $\square$

In [18] it was shown that the removal of one point from 2^{ω_1} gives an irreducible space. However, this space is not a D-space, as shown by the following lemma. Note that every nonempty open subset of this space has the same cardinality.

Lemma 1.2. *The space 2^{ω_1} with one point removed contains a closed copy of ω_1.*

Proof. For simplicity, let the removed point be the constant 0 function. For each $\alpha < \omega_1$, let x_α be $(\alpha \times \{0\}) \cup ((\omega_1 \setminus \alpha) \times \{1\})$. The subspace $\{x_\alpha : \alpha < \omega_1\}$ is closed. Furthermore, if for each $\beta < \omega_1$, we let $O_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 0\}$ and $I_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 1\}$, then the O_β's and I_β's generate the ω_1-topology on $\{x_\alpha : \alpha < \omega_1\}$. □

2. Lindelöfness and Martin's Axiom

Our second section concerns Lindelöf non-D-spaces and Martin's Axiom (MA). Recall that Martin's Axiom is the statement that if P is a partial order without uncountable antichains, and $\mathcal{D}$ is a collection of dense subsets of P such that $|\mathcal{D}| < \mathfrak{c}$ (where $\mathfrak{c}$ denotes the cardinality of the continuum), then there is a filter $G \subseteq P$ intersecting each element of $\mathcal{D}$ (see [17], for instance). The *covering number for the meager ideal* ($cov(\mathcal{M})$) is the smallest cardinality of a family of meager sets of reals whose union is all of $\mathbb{R}$ (see [5], for instance); restated, it is the smallest cardinality of a collection $\mathcal{D}$ consisting of dense subsets of the partial order $(\omega^{<\omega}, \subseteq)$ with the property that no filter intersects every member of $\mathcal{D}$. The Baire Category Theorem implies that $cov(\mathcal{M}) \geq \aleph_1$. Martin's Axiom (indeed, its restriction to Cohen forcing) implies that $cov(\mathcal{M}) = \mathfrak{c}$.

We first prove that there are no Lindelöf non-D-spaces of cardinality less than $cov(\mathcal{M})$. One easy consequence is that one cannot prove, assuming only ZFC, that there is such space of cardinality $\aleph_1$.

We begin with the following.

Lemma 2.1. *If X is a Lindelöf space and N is an open neighborhood assignment on X, then there is a countable $Y \subseteq X$ such that for every finite $a \subseteq Y$ and every $x \in X \setminus N[a]$ there is a $y \in Y \setminus N[a]$ such that $x \in N(y)$.*

Proof. We find countable sets $Y_i \subseteq X$ $(i < \omega)$, and let $\langle a_i : i < \omega\rangle$ be a listing of all the finite subsets of $\bigcup_{i<\omega} Y_i$, such that each $a_i \subseteq \bigcup_{j\leq i} Y_j$. Let Y_0 be any countable subset of X such that $N[Y_0] = X$. Given a_i, let $\mathcal{U}_i$ be a countable subcover of the open cover of X given by the restriction of N to $a_i \cup (X \setminus N[a_i])$, and let Y_{i+1} be a countable subset of X such that $\mathcal{U}_i = \{N(x) \mid x \in Y_{i+1}\}$. Then $Y = \bigcup_{i<\omega} Y_i$ is as desired. □

Theorem 2.2. *If X is a Lindelöf space and $|X| < cov(\mathcal{M})$, then X is a D-space.*

Proof. Let N be an ONA on X. Let $Y = \langle y_i : i < \omega\rangle$ be as in Lemma 2.1, and consider the set A of $a \in 2^\omega$ such that for each $i \in a^{-1}(1)$,

$$y_i \notin N[\{y_j : j \in i \cap a^{-1}(1)\}].$$

Assuming that no finite subset of the range of N covers X, A is a perfect subset of 2^ω, and for each $x \in X$ the set of $a \in A$ with $x \notin N[\{y_i : i \in a\}]$ is nowhere dense in A. Since $|X| < cov(\mathcal{M})$, there is an $a \in A$ such that $N[a] = X$. □

The assumptions of Theorem 2.2 are implied by MA(Cohen forcing) when $|X| < \mathfrak{c}$ (see [5]).

Corollary 2.3 (MA(Cohen forcing))**.** *If X is a Lindelöf space such that $|X| < 2^\omega$, then X is a D-space.*

Corollary 2.4 (MA(Cohen forcing))**.** *If X is a hereditary Lindelöf space which is not a D-space, then $|X| = 2^\omega$.*

Proof. This is immediate, since if X is a hereditary Lindelöf space then $|X| \leq 2^\omega$, by a result of de Groot (see [15]). □

Let (X, τ) be a topological space. Let $f : \omega^{<\omega} \longrightarrow X \times \tau$ be a function. If $s \in \omega^{<\omega}$ and $f(s) = (x, V)$, then we write $f_X(s) = x$ and $f_\tau(s) = V$.

The idea for the next lemma is the following: we will construct an ω-tree using the Y given by the previous lemma. The successors of every element of the tree will be all the points of Y that are "not yet covered" by our construction. At the same time we will assure that every finite subset of Y that is not yet covered can be added to the tree in finitely many steps.

Lemma 2.5. *Let (X, τ) be a Lindelöf space and let N be an open neighborhood assignment on X such that no finite subset of the range of N covers X. Then there is $f : \omega^{<\omega} \setminus \{\emptyset\} \longrightarrow X \times \tau$ such that:*

(i) if $s \in \omega^{<\omega} \setminus \{\emptyset\}$ then $f_\tau(s) \subseteq N(f_X(s))$;
(ii) if r is a branch of $\omega^{<\omega}$, then $\{f_X(s) : s \in r\}$ is closed discrete in $\bigcup\{f_\tau(s) : s \in r\}$;
(iii) if $C \subseteq X$ is compact, then $D_C = \{s \in \omega^{<\omega} : C \subseteq \bigcup_{t \leq s} f_\tau(t)\}$ is dense in $\omega^{<\omega}$.

Proof. Let Y be as given by Lemma 2.1. We will define $f : \omega^{<\omega} \setminus \{\emptyset\} \longrightarrow Y \times \tau$ by recursion on the length of s in such a way that:

(a) if $s \in \omega^{<\omega}$ then for every $n \in \omega$ and every nonzero $k \leq |s|$, $f_X(s^\frown n) \notin f_\tau(s \restriction k)$;
(b) if $s \in \omega^{<\omega}$ then $f_\tau(s) = N(f_X(s)) \setminus F$ where F is a finite subset of $Y \setminus \{f_X(s)\}$;
(c) for every $s \in \omega^{<\omega}$, if $y \in Y \setminus \bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\}$, then there is an $n \in \omega$ such that $y = f_X(s^\frown n)$;
(d) if $y = f_X(s^\frown n)$ for some $s \in \omega^{<\omega}$ and $n \in \omega$, then for each finite $F \subseteq (Y \cap N(y)) \setminus \{y\}$ there is a $k \in \omega$ such that $f(s^\frown k) = (y, N(y) \setminus F)$;

Note that we can make this construction since Y is countable and so is $[Y]^{<\omega}$.

First we will show that if r is a branch of $\omega^{<\omega} \setminus \{\emptyset\}$, then $\{f_X(s) : s \in r\}$ has no accumulation points in $\bigcup\{f_\tau(s) : s \in r\}$. Let $x \in \bigcup\{f_\tau(s) : s \in r\}$. We will show that it is not an accumulation point of $\{f_X(s) : s \in r\}$. Let $s \in r$ such that $x \in f_\tau(s)$. Note that $f_X(t) \notin f_\tau(s)$ for every $t \in r$, $t > s$. Then x is separated from these points and, since there are only finitely many points more in r, we have that x is not an accumulation point.

Note that, by Lemma 2.1 and condition (c), we have that, for every $s \in \omega^{<\omega} \setminus \{\emptyset\}$,

$$\bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\} \cup \bigcup_{n \in \omega} f_\tau(s^\frown n) = X.$$

For each $C \subseteq X$, let D_C denote the set of $s \in \omega^\omega \setminus \{\emptyset\}$ such that C is contained in $\bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\}$. We will show that when C is compact, D_C is dense in $\omega^\omega \setminus \{\emptyset\}$. Fix $s \in \omega^{<\omega}$ and suppose $s \notin D_C$. Then $C \setminus \bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\}$ is a compact subset covered by the family $(f_\tau(s^\frown n))_{n\in\omega}$. Then, there are $k_1, ..., k_n \in \omega$ such that

$$C \setminus \bigcup\{f_\tau(s{\restriction}k) : 0 < k \leq |s|\} \subseteq f_\tau(s^\frown k_1) \cup \cdots \cup f_\tau(s^\frown k_n).$$

We can suppose that $f_X(s^\frown k_i) \notin f_\tau(s^\frown k_j)$ for all $1 \leq i < j \leq n$ by property (d). Thus, we can choose $p_1, ..., p_n \in \omega$ such that $f(s^\frown p_1) = f(s^\frown k_1)$, $f(s^\frown p_1^\frown p_2) = f(s^\frown k_2)$, ..., $f(s^\frown p_1^\frown p_2^\frown \cdots ^\frown p_n) = f(s^\frown k_n)$, by property (c). Note that $C \subseteq \bigcup_{t \leq s^\frown p_1^\frown p_2^\frown \cdots ^\frown p_n} f_\tau(t)$. □

It was already known that every σ-compact space is a D-space (see [6], for instance). Lemma 2.5 allows us to improve this result.

Theorem 2.6. *Every Lindelöf space which is a union of fewer than $cov(\mathcal{M})$ many compact spaces is a D-space.*

Proof. Let X be a Lindelöf space. If X is compact, then it is a D-space, so suppose that X is not compact. Let N be an ONA on X. By shrinking the values taken by N if necessary, we may suppose that no finite subset of the range of N covers X. Suppose that κ is a cardinal less than $cov(\mathcal{M})$, and that $X = \bigcup_{\xi<\kappa} C_\xi$, where each C_ξ is compact. Let $f : \omega^{<\omega} \longrightarrow X \times \tau$ be the function given by Lemma 2.5. Note that for every $\xi < \kappa$ we have that D_{C_ξ} (as defined in the proof of Lemma 2.5) is dense in $\omega^{<\omega}$. Then there is a branch r of $\omega^{<\omega}$ such that $r \cap D_{C_\xi} \neq \emptyset$ for each $\xi < \kappa$. Thus $\bigcup_{s\in r} f_\tau(s) \supset \bigcup_{\xi<\kappa} C_\xi = X$. Since $\{f_X(s) : s \in r\}$ is closed discrete in $\bigcup_{s\in r} f_\tau(s)$, we have that $\{f_X(s) : s \in r\}$ is closed discrete in X. □

3. Forcing with trees of height ω

The results of the previous section suggest that some basic facts about D-spaces may be independent of ZFC. While we do not have such a result, we present in this section two facts about D-spaces and forcing which may be of some use. These facts concern forcing with trees of height ω, and apply the approach of the previous section. In order to make the forcing order match the tree order, we adopt the convention that trees grow downwards, so $s \leq t$ means that s is greater than t in the usual tree order.

Theorem 3.1. *If X is a D-space and T is a tree of height ω, then X remains a D-space after forcing with T.*

Proof. Let $\dot{N}$ be a T-name for an ONA on X, and let $\mathcal{B}$ be the set of open subsets of X which are forced by some condition to be in the range of the realization of $\dot{N}$ (we may assume that the range of the realization of $\dot{N}$ will consist of sets in the ground model). Let $\langle p_\alpha : \alpha < \kappa \rangle$ be a wellordering of the elements of T such that $\alpha \leq \beta$ whenever $p_\alpha \geq p_\beta$. We define recursively on α closed discrete sets D_α ($\alpha < \kappa$) and functions $f_\alpha \colon D_\alpha \to T$ and $h_\alpha \colon D_\alpha \to \mathcal{B}$ such that, letting Y_α be the set of x in any D_β ($\beta < \alpha$) such that $f_\beta(x) \geq p_\alpha$:

- for all x in any D_α, $f_\alpha(x) \leq p_\alpha$ and $f_\alpha(x) \Vdash \dot{N}(\check{x}) = h_\alpha(x)\check{}$;
- for all $\alpha < \kappa$ and for all $y \in X$, either there exist $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$ such that $y \in h_\beta(x)$ or there exists an $x \in D_\alpha$ such that $y \in h_\alpha(x)$;
- if $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$, then $h_\beta(x) \cap D_\alpha = \emptyset$.

(Note that since $f_\beta(x) \leq p_\beta$ for each $\beta < \kappa$ and each $x \in X$, $p_\beta \geq p_\alpha$ whenever $D_\beta \cap Y_\alpha$ is nonempty; in particular there are only finitely many such β, so Y_α is closed discrete.)

Supposing that we have constructed D_β, f_β and h_β for all $\beta < \alpha$, let

$$E_\alpha = \bigcup \{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA N_α as follows. For each $x \in E_\alpha$, let $N_\alpha(x) = E_\alpha$. For each $x \in X \setminus E_\alpha$, pick a condition $p(x) \leq p_\alpha$ and an element $B(x) \in \mathcal{B}$ such that $p(x) \Vdash \dot{N}(\check{x}) = B(x)\check{}$, and let $N_\alpha(x) = B(x)$. Then there is a closed discrete set D^*_α such that $N_\alpha[D^*_\alpha] = X$. Let $D_\alpha = D^*_\alpha \setminus E_\alpha$. For each $x \in D_\alpha$, let $f_\alpha(x) = p(x)$ and let $h_\alpha(x) = B(x)$. This completes the construction.

Let g be a V-generic path through T. For each $\alpha \in \kappa$, let C_α be the set of $x \in D_\alpha$ such that $f_\alpha(x) \in g$. Let

$$C = \bigcup \{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity $\dot{N}_g[C] = X$. We will be done once we show that C is closed discrete.

Pick a point y in X. There is a $x \in C_\beta$ for some $\beta \in \kappa$ such that $y \in \dot{N}_g(x)$. Since $f_\beta(x) \in g$, $\dot{N}_g(x) = h_\beta(x)$. Fix $\gamma < \kappa$ such that $p_\gamma \in g$, $\gamma > \beta$ and $p_\gamma \leq f_\beta(x)$. Since $D_\alpha \cap h_\beta(x) = \emptyset$ for all $\alpha > \beta$ with $p_\alpha \leq f_\beta(x)$, y is not in the closure of

$$\bigcup \{D_\alpha \mid \gamma \leq \alpha < \kappa, p_\alpha \in g\},$$

which contains $\bigcup\{C_\alpha : \gamma \leq \alpha < \kappa\}$. On the other hand,

$$C \subseteq \bigcup\{D_\alpha : p_\alpha > p_\gamma\} \cup \bigcup\{C_\alpha \mid \gamma \leq \alpha < \kappa\}.$$

Since $\bigcup\{D_\alpha : p_\alpha > p_\gamma\}$ is a finite union of closed discrete sets, y is not in the closure of $\bigcup\{D_\alpha : p_\alpha > p_\gamma\} \setminus \{y\}$, either, which shows that C is closed discrete. $\square$

A similar argument shows the following result, where we start with a Lindelöf space in the ground model. If T is a tree and S is a subset of T, we say that S can be refined to an antichain if there is a function $a \colon S \to T$ such that $a(s) \leq s$ for all $s \in S$, and such that the range of S is an antichain.

Theorem 3.2. *If X is a Lindelöf space, T is a tree of height ω such that every countable subset of T can be refined to an antichain, then X is a D-space after forcing with T.*

Proof. Let $\dot{N}$ be a T-name for an ONA on X, and let $\mathcal{B}$ be the set of open subsets of X which are forced by some condition to be in the range of the realization of $\dot{N}$. Let $\langle p_\alpha : \alpha < \kappa\rangle$ be a wellordering of the elements of T such that $\alpha \leq \beta$ whenever $p_\alpha \geq p_\beta$. We define recursively on α countable sets $D_\alpha \subseteq X$ $(\alpha < \kappa)$ and functions $f_\alpha \colon D_\alpha \to T$ and $h_\alpha \colon D_\alpha \to \mathcal{B}$ such that, letting Y_α be the set of x in any D_β $(\beta < \alpha)$ such that $f_\beta(x) \geq p_\alpha$:

- for all x in any D_α, $f_\alpha(x) \leq p_\alpha$ and $f_\alpha(x) \Vdash \dot{N}(\check{x}) = h_\alpha(x)\check{}$;
- for all $\alpha < \kappa$ and for all $y \in X$, either there exist $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$ such that $y \in h_\beta(x)$ or there exists an $x \in D_\alpha$ such that $y \in h_\alpha(x)$;
- if $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$, then $h_\beta(x) \cap D_\alpha = \emptyset$;
- the range of each f_α is an antichain.

Supposing that we have constructed D_β, f_β and h_β for all $\beta < \alpha$, let

$$E_\alpha = \bigcup\{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA N_α as follows. For each $x \in E_\alpha$, let $N_\alpha(x) = E_\alpha$. For each $x \in X \setminus E_\alpha$, pick a condition $p(x) \leq p_\alpha$ and an element $B(x) \in \mathcal{B}$ such that $p(x) \Vdash \dot{N}(\check{x}) = B(x)\check{}$, and let $N_\alpha(x) = B(x)$. Then there is a countable set D^*_α such that $N_\alpha[D^*_\alpha] = X$. Let $D_\alpha = D^*_\alpha \setminus E_\alpha$. For each $x \in D_\alpha$, let $f_\alpha(x)$ be a condition below $p(x)$ in such a way that the range of f_α is an antichain, and let $h_\alpha(x) = B(x)$. This completes the construction.

Let g be a V-generic path through T. For each $\alpha \in \kappa$, let C_α be the set of $x \in D_\alpha$ such that $f_\alpha(x) \in g$. Let

$$C = \bigcup\{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity $\dot{N}_g[C] = X$. We will be done once we show that C is closed discrete.

Pick a point y in X. There is a $x \in C_\beta$ for some $\beta \in \kappa$ such that $y \in \dot{N}_g(x)$. Since $f_\beta(x) \in g$, $\dot{N}_g(x) = h_\beta(x)$. Fix $\gamma < \kappa$ such that $p_\gamma \in g$, $\gamma > \beta$ and $p_\gamma \leq f_\beta(x)$. Since $D_\alpha \cap h_\beta(x) = \emptyset$ for all $\alpha > \beta$ with $p_\alpha \leq f_\beta(x)$, y is not in the closure of

$$\bigcup\{D_\alpha \mid \gamma \leq \alpha < \kappa, p_\alpha \in g\},$$

which contains $\bigcup\{C_\alpha : \gamma \leq \alpha < \kappa\}$. On the other hand, letting

$$Z = \bigcup\{\{z \in D_\alpha \mid f_\alpha(z) \in g\} : \alpha < \gamma\},$$

$$C = Z \cup \bigcup\{C_\alpha \mid \gamma \leq \alpha < \kappa\}.$$

Since Z is a finite set, y is not in the closure of $Z \setminus \{y\}$, either, which shows that C is closed discrete. □

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