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A. Gryzlov, E. Bastrykov and R. Golovastov

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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ON BELL'S COMPACTIFICATION OF N

A. GRYZLOV, E. BASTRYKOV, AND R. GOLOVASTOV

Abstract. We consider Bell's compactification BN of N, and examine some properties of subsets of BN.

1. Introduction

M. G. Bell [1] constructed a compactification BN of a countable discrete space N with ccc non-separable remainder. The compactification with such properties is very interesting itself, widely used in the theory of βN, particular for a construction of some non p-points (see [2, 3]). We consider further properties of BN and compare them with the properties of βN.

Recall that the closure of any infinite subset of $N \subseteq \beta N$ is homeomorphic to βN. For BN the situation is the following:

– On the one hand, there are infinite subsets $A \subseteq N \subseteq BN$ whose closures are homeomorphic to βN (Theorem 3.6);

– On the other hand there are infinite subsets of N, which are convergent sequences in BN (Theorem 3.9).

Moreover, remainders of these subsets form π-nets of $BN \setminus N$ (Theorem 3.10). Using these results we show some basic cardinal invariants of BN.

Recall that in the space βN the remainders of two almost disjoint subsets of N are disjoint. The situation in BN is more complicated (Proposition 3.11).

Finally, we describe a σ-n-linked base of $BN \setminus N$ with some additional properties (Theorem 3.12).

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2. Preliminaries

The space βN is the Stone-Čech compactification of a countable discrete space.

By $[A]$ we denote the closure of a set A.

For a compactification BN of N and for $A \subseteq N \subseteq BN$ we denote

$$A^* = [A] \setminus A.$$

Recall some properties of βN.

Proposition 2.1. *N contains an almost disjoint system of subsets $\lambda = \{A\}$ of the power 2^ω.*

Theorem 2.2.

(1) $|\beta N| = 2^{2^\omega}$;
(2) $\mathrm{w}(\beta N) = \mathrm{w}(\beta N \setminus N) = 2^\omega$;
(3) $c(\beta N \setminus N) = 2^\omega$;
(4) $t(\beta N) = 2^\omega$;
(5) $\chi(\beta N) = 2^\omega$.

Construction of Bell's compactification [1]**:**

Let $P = \{f \in \omega^\omega : 0 \leq f(n) \leq n+1 \text{ for all } n \in \omega\}$ and

$$N = \{f|_n : f \in P,\ n \in \omega\}.$$

Define $T = \{\pi \in N^\omega : \operatorname{dom} \pi(n) = n+1 \text{ for all } n \in \omega\}$.

For every $s \in N$ let $C_s = \{t \in N : t|_{\operatorname{dom} s} = s\}$.

For $\pi \in T$ let $C_\pi = \cup\{C_{\pi(n)} : n \in \omega\}$.

Let B be a Boolean algebra, generated by

$$B' = \{C_\pi : \pi \in T\} \cup \{N \setminus C_\pi : \pi \in T\}.$$

Let $B'' = \{U : U \in B, |U| = \omega\}$.

Note that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq B$.

Denote BN the Stone space of B. BN is a compactification of the countable discrete space $N = \{f|_n : f \in P,\ n \in \omega\}$. We will denote points of N as $f|_n$, s, t.

Note, that the countable set $N = \{f|_n : f \in P,\ n \in \omega\}$ is a partially ordered set with the order: $s \leq t$ if t extends s for s, $t \in N$.

Theorem 2.3. [1] *B'' is a countable union of 2-linked families.*

Theorem 2.3 implies the first half of the following Corollary:

Corollary 2.4. [1] *The space* $N^* = BN \setminus N$ *satisfies Suslin condition but is not separable.*

Theorem 2.5. [3] *For any* $m \in \omega$ *there is a system of sets* B_m, $B_m \subseteq B''$ *such that*

(1) $|N \setminus \{U_1 \cup \ldots \cup U_m\}| = \omega$ *for any set* $\{U_1, \ldots, U_m\} \subset B_m$;
(2) *for any countable set of points* $\{p_k : k \in \omega\} \subset N^*$ *there is* $U \in B_m$, *such that* $\{p_k : k \in \omega\} \subseteq [U]$.

Definition 2.6. [7] A family of sets $\lambda = \{M_\alpha\}$ is called n-linked if $|\cap \{M_{\alpha_i} : i \leq n\}| \geq \omega$ for every $\{M_{\alpha_1}, \ldots, M_{\alpha_n}\} \subseteq \lambda$.

3. Main Results

For $\pi \in T$ and $M \subseteq \omega$ let us denote

$$C_{\pi|M} = \cup\{C_{\pi(n)} : n \in M\}.$$

Lemma 3.1. $C_{\pi|M}$ *is the element of* B.

Proof. Define

$$M_0 = \{n \in M : \pi(n)(0) = 0\},$$

$$M_1 = \{n \in M : \pi(n)(0) = 1\}.$$

Note, that $M_0 \cap M_1 = \emptyset$ and $M_0 \cup M_1 = M$.

Let $f_0 \in P$ be a function such that $f_0(n) = 0$ for all $n \in \omega$, and $f_1 \in P$ be a function such that $f_1(n) = 1$ for all $n \in \omega$. We define π_0 and π_1, as following

$$\pi_0(n) = \begin{cases} \pi(n) & \text{for} \quad n \in M_0 \\ f_1|_{n+1} & \text{for} \quad n \notin M_0 \end{cases}$$

$$\pi_1(n) = \begin{cases} \pi(n) & \text{for} \quad n \in M_1 \\ f_0|_{n+1} & \text{for} \quad n \notin M_1 \end{cases}$$

Denote:

$$C_0 = \{s \in N : s(0) = 0\}, \quad C_1 = \{s \in N : s(0) = 1\},$$

then $C_0, C_1, C_{\pi_0}, C_{\pi_1} \in B$.

It is easy to see, that $C_{\pi|M} = (C_{\pi_0} \cap C_0) \cup (C_{\pi_1} \cap C_1)$. □

Lemma 3.2. *For a family* $\{C_{\pi_i|M_i} : i \leq n\}$ $(n \in \omega)$ *the following is true:*

$$\bigcap_{i=1}^{n} C_{\pi_i|M_i} = \bigcup_{i=1}^{n} C_{\pi_i|M'_i}, \text{ where } M'_i \subseteq M_i \text{ for all } i \leq n.$$

Proof. Let us prove for $n = 2$. We construct M'_1 and M'_2 such that,

$$C_{\pi_1|M_1} \cap C_{\pi_2|M_2} = C_{\pi_1|M'_1} \cup C_{\pi_2|M'_2}.$$

Denote

$$M'_1 = \{k \in M_1 : \text{there is } m \in M_2 \text{ such that } C_{\pi_1(k)} \subseteq C_{\pi_2(m)}\}$$

$$M'_2 = \{k \in M_2 : \text{there is } m \in M_1 \text{ such that } C_{\pi_2(k)} \subseteq C_{\pi_1(m)}\}.$$

Note that for any s and t from N, C_s either contains C_t, or is included to C_t (in particular, they are equal), or they are disjoint. From this it follows that

$$C_{\pi_1|M'_1} \cup C_{\pi_2|M'_2} = C_{\pi_1|M_1} \cap C_{\pi_2|M_2}.$$

Now by induction we get

$$(\bigcap_{i=1}^{n} C_{\pi_i|M_i}) \cap C_{\pi_{n+1}|M_{n+1}} = (\bigcup_{i=1}^{n} C_{\pi_i|M'_i}) \cap C_{\pi_{n+1}|M_{n+1}} =$$

$$\bigcup_{i=1}^{n}(C_{\pi_i|M'_i} \cap C_{\pi_{n+1}|M_{n+1}}) = \bigcup_{i=1}^{n}(C_{\pi_i|M''_i} \cup C_{\pi_{n+1}|M^i_{n+1}}) = \bigcup_{i=1}^{n+1} C_{\pi_i|M''_i}$$

where $M''_{n+1} = \bigcup\limits_{i=1}^{n} M^i_{n+1}$. □

Corollary 3.3. *Let* $x \in BN \setminus N$ *and* $x \in [\bigcap\limits_{i=1}^{n} C_{\pi_i|M_i}]$, *then there are* π_{i_0} *and* $M'_{i_0} \subseteq M_{i_0}$, *such that* $x \in [C_{\pi_{i_0}|M'_{i_0}}] \subseteq [\bigcap\limits_{i=1}^{n} C_{\pi_i|M_i}]$.

Denote

$$\Gamma = \{C_{\pi|M} \setminus \bigcup_{i=1}^{n} C_{\pi_i} : \pi,\, \pi_i \in T,\ M \subseteq \omega,\ n \in \omega\}.$$

Now by Lemma 3.1 and Corollary 3.3 we get:

Theorem 3.4. $\widetilde{B} = \{U^* : U \in \Gamma,\ |U| = \omega\}$ *is a base of the space* N^*.

Recall that in βN the closure of every infinite subset of N is homeomorphic to βN and is a clopen set. For BN there are infinite subsets of N whose closures are homeomorphic to βN and infinite subsets of N which are convergent sequences.

Lemma 3.5. *Let $\{C_{s_i} : i \in \omega\}$ be a disjoint family of subsets, $\operatorname{dom} s_i \neq \operatorname{dom} s_j$ for all $i \neq j$, and $X = \{x_i : i \in \omega\}$ such that $x_i \in [C_{s_i}]$. Then $[X]$ is homeomorphic to βN.*

Proof. Let us consider a family $\lambda = \{C_{s_i} : i \in \omega\}$.

For an ultrafilter $\xi \in \beta N \setminus N$ and $F \in \xi$ denote:

$$W_F = [\cup\{C_{s_i} : i \in F\}],$$
$$\lambda_\xi = \{W_F : F \in \xi\},$$
$$L_\xi = \cap\{[W_F] : W_F \in \lambda_\xi\}.$$

It is easy to see that the following is true:

(1) $L_\xi \cap L_\eta = \emptyset$ for $\xi \neq \eta$;
(2) $\{X \cap W_F : f \in \xi\}$ is an ultrafilter on X;
(3) $|L_\xi \cap [X]| = 1$.

Let $x_\xi = \cap\{[X \cap W_F] : F \in \xi\}$. From the construction it follows that $X \cup \{x_\xi : \xi \in \beta N \setminus N\} = [X]$ is homeomorphic to βN. □

Note, that X^* is a nowhere dense in N^* since $c(N^*) = 2^\omega$.

If we consider points $x_i \in C_{s_i}$ $(i \in \omega)$ in Lemma 3.5, we get:

Theorem 3.6. *There is an infinite $A \subseteq N$ such that $[A]$ is homeomorphic to βN.*

Theorem 3.7. *If $X = \{x_i : i \in \omega\}$ is a discrete subset of N^* such that $x_i \in C^*_{s_i}$, where $C^*_{s_i} \cap C^*_{s_j} = \emptyset$ for $i \neq j$. Then $[X]$ is homeomorphic to βN.*

The proof follows from the fact that

$$C^*_s = \cup\{C^*_t : \operatorname{dom} t = \operatorname{dom} s + 1,\ s \leq t\} \text{ for any } C^*_s.$$

From these theorems and properties of βN (Theorem 2.2) we get:

Corollary 3.8.

(1) $\mathrm{w}(BN) = 2^\omega$;
(2) $s(BN) = 2^\omega$;
(3) $t(BN) = 2^\omega$;
(4) $|BN| = 2^{2^\omega}$.

Theorem 3.9. *Let $A = \{s_i : i \in \omega\}$ be an infinite chain of N. Then A is a convergent sequence in BN.*

Proof. Let $A = \{s_i : i \in \omega\}$ be an infinite chain in N, i. e. s_{i+1} extends s_i, $\operatorname{dom} s_i < \operatorname{dom} s_{i+1}$ for $i \in \omega$. Let $x \in [A] \setminus A$ and

$$O_x = [C_{\pi|M} \setminus \bigcup_{i=1}^{n} C_{\pi_i}]$$

be a basic neighbourhood of x.

There is $s_{i_0} \in A$ such that $s_{i_0} \in O_x$, then $s_{i_0} \in C_{\pi|M}$ and therefore $s_i \in C_{\pi|M}$ for all $i \geq i_0$. On the other hand we have:

$$\bigcup_{i=1}^{n} C_{\pi_i} \cap A = \emptyset,$$

otherwise $\bigcup_{i=1}^{n} C_{\pi_i}$ contains all but finitely many points of A.

So, x is a limit of the convergent sequence $A = \{s_i : i \in \omega\}$. □

Theorem 3.10. *Let*

$$Q = \{x : x \text{ is a limit of a convergent sequence of points of } N\},$$
$$\mu = \{A^* : A \subseteq N,\ [A] \text{ is homeomorphic to } \beta N\}.$$

Then Q is a dense set in N and μ is a π-net of N^*.

Proof. Let $V = C_{\pi|M} \setminus \bigcup_{i=1}^{m} C_{\pi_i}$, be an element of Γ, $|V| = \omega$. By induction we construct two sequences $\{s_k : k \in \omega\}$ and $\{t_k : k \in \omega\}$ in V such that:

(1) $\{s_k : k \in \omega\}$ is a chain in N;
(2) $\{t_k : k \in \omega\}$ is an anti-chain in N and $\operatorname{dom} t_k \neq \operatorname{dom} t_\ell$ for $k \neq \ell$;
(3) t_{k+1} extend s_k for all $k \in \omega$.

Let n_0 and $s \in N$ be such that $n_0 \geq m$ and $\operatorname{dom} s = n_0 + 1$, $s \in C_{\pi|M} \setminus \bigcup_{i=1}^{m} C_{\pi_i}$. Denote $s_0 = t_0 = s$.

Then we define $\{s_k : k \leq \ell\}$ and $\{t_k : k \leq \ell\}$, satisfying conditions (1)–(3). Since $\ell \geq n_0 \geq m$ there are at least two extensions of s_ℓ belonging to $C_{\pi|M} \setminus \bigcup_{i=1}^{m} C_{\pi_i}$. We define one of them as $s_{\ell+1}$ and another as $t_{\ell+1}$.

So we get $\{s_k : k \in \omega\}$ and $\{t_k : k \in \omega\}$. By the construction we have $\{s_k : k \in \omega\} \subseteq V$, $\{s_k : k \in \omega\}$ is a convergent sequence and $\lim\limits_{k\to\infty} s_k \in V^*$, $\{t_k : k \in \omega\} \subseteq V$, $[\{t_k : k \in \omega\}]$ is homeomorphic to βN and $(\{t_k : k \in \omega\})^* \subseteq V^*$. □

In the space βN remainders of any two almost disjoint subsets are disjoint. But for BN the situation is quite different.

Proposition 3.11. *There are families of almost disjoint subsets N:*

(1) $\lambda_1 = \{A_\alpha : \alpha \in 2^\omega\}$ *such that* $[A_\alpha]$ *homeomorphic to* βN *for all* $\alpha \in 2^\omega$, $A^*_\alpha \cap A^*_\beta = \emptyset$ *for* $\alpha \neq \beta$;

(2) $\lambda_2 = \{A_\alpha : \alpha \in 2^\omega\}$ *such that* $|A^*_\alpha| = 1$ *and* $A^*_\alpha = A^*_\beta$ *for all* $\alpha, \beta \in 2^\omega$;

(2′) $\lambda'_2 = \{A_\alpha : \alpha \in 2^\omega\}$ *such that* $|A^*_\alpha| = 1$ *and* $A^*_\alpha \cap A^*_\beta = \emptyset$ *for* $\alpha \neq \beta$;

(3) $\lambda_3 = \{A_\alpha : \alpha \in 2^\omega\}$ *such that* $A^*_\alpha = N^*$ *for all* $\alpha \in 2^\omega$.

Proof. Let $\theta = \{F_\alpha : \alpha \in 2^\omega\}$ be a family of infinite almost disjoint sets.

(1) Let $A_1 = \{s_i : i \in \omega\}$ be a set such that $[A_1]$ homeomorphic to βN. Denote $A_1(F_\alpha) = \{s_i : i \in F_\alpha\}$.

The family $\lambda_1 = \{A_1(F_\alpha) : \alpha \in 2^\omega\}$ is as required in (1).

(2) By the same way for a convergent sequence $A_2 = \{s_i : i \in \omega\}$ we can construct a family $\lambda_2 = \{A_2(F_\alpha) : \alpha \in 2^\omega\}$ satisfing (2).

(2′) To prove this fact we can consider a almost disjoint family λ'_2 of countable chains in N, such that $|\lambda'_2| = 2^\omega$.

(3) Let $A_\alpha = \{s \in N : \operatorname{dom} s = n \text{ for all } n \in F_\alpha\}$. It is easy to see that $\lambda_3 = \{A_\alpha : \alpha \in 2^\omega\}$ is as required. □

Now we prove the theorem which combines Theorems 2.3 and 2.5.

Let $n \in \omega$, $s \in N$ be such that $\operatorname{dom} s \geq n$. Denote

$$\Gamma(s) = \{C_{\pi|M} \setminus \bigcup_{i=1}^{m} C_{\pi_i} : \pi(\min M) = s \text{ and } s \notin \bigcup_{i=1}^{m} C_{\pi_i} \text{ and } mn < \operatorname{dom} s + 1\}.$$

Theorem 3.12.

(1) *Let $n \in \omega$ and $s \in N$ such that* $\operatorname{dom} s \geq n$. *Then for* $\Gamma(s)$ *the following is true:*

(a) $\Gamma(s)$ *is a n-linked family;*
(b) *the family of complements of elements of $\Gamma(s)$ is a n-linked family;*
(c) *for every countable set of points $\{p_i : i \in \omega\} \subseteq N^*$ there is $U \in \Gamma(s)$, such that $\{p_i : i \in \omega\} \subseteq U^*$.*

(2) *For every $n \in \omega$ a family*

$$\Gamma_n = \{[U] \setminus N : U \in \cup\{\Gamma(s) : \operatorname{dom} s \geq n\}\}$$

is a base of N^*.

Proof. (1a) We modify the Bell's proof (see [1]). Let $n \in \omega$ and s be such that $\operatorname{dom} s \geq n$. Let $U_1, \ldots, U_n \in \Gamma(s)$. Then

$$s \in U_j = C_{\pi_j|M_j} \setminus \bigcup_{i=1}^{m_j} C_{\pi_i^j} \quad (j = 1, \ldots, n).$$

We construct $h \in P$ such that $\{h|_k : k \geq \operatorname{dom} s\} \subseteq U_1 \cap \ldots \cap U_n$ by induction on $k \geq \operatorname{dom} s$. Let $k|_{\operatorname{dom} s} = s \in U_1 \cap \ldots \cap U_n$. If $h|_k \in U_1 \cap \ldots \cap U_n$ for $k \geq \operatorname{dom} s$ is defined, we can define $h|_{k+1} \notin \{\pi_i^j(k) : 1 \leq i \leq m_j,\ j = 1, \ldots, n\}$ since

$$|\{(i, j) : 1 \leq j \leq n,\ 1 \leq i \leq m_j\}| \leq n \cdot \max m_j < k + 2$$

and there are $k + 2$ many extensions of $h|_k$ on $k + 1$.

(1b) Let $U_j = C_{\pi_j|M_j} \setminus \bigcup_{i=1}^{m_j} C_{\pi_i^j} \in \Gamma(s)$ $(j = 1, \ldots, n)$. We construct $h \in P$ such that $\{h|_k : k \geq \operatorname{dom} s\} \subseteq N \setminus \bigcup_{j=1}^{n} C_{\pi_j|M_j}$.

Let $h|_{\operatorname{dom} s} \neq s$, therefore $h|_{\operatorname{dom} s} \notin \bigcup_{j=1}^{n} C_{\pi_j|M_j}$. If $h|_k \notin \bigcup_{j=1}^{n} C_{\pi_j|M_j}$ for $k \geq \operatorname{dom} s$ is defined, we can define $h|_{k+1} \notin \bigcup_{j=1}^{n} C_{\pi_j|M_j}$ since

$$|\{\pi_j(k) : j = 1, \ldots, n\}| \leq n \leq \operatorname{dom} s < k + 2$$

and there are $k + 2$ many extensions of $h|_k$ on $k + 1$.

It is easy to see, that $\Gamma(s)$ satisfy condition (1c).

(2) Let $x \in N^*$, $O_x = C_{\pi|M} \setminus (\bigcup_{i=1}^{m} C_{\pi_i})$ and ℓ_0 be such that $mn < \ell_0 + 1$ and $\ell_0 \geq n$. There is s_0 such that $\operatorname{dom} s_0 = \ell_0$

and $x \in [C_{s_0}]$. We have

$$C_{s_0} \cap \left(C_{\pi|M} \setminus \bigcup_{i=1}^{m} C_{\pi_i} \right) = C_{\pi|M'} \setminus \bigcup_{i=1}^{m} C_{\pi_i},$$

where $M' = \{\ell : \ell \in M \text{ and } \pi(\ell) \text{ is extension of } s_0\}$. Obviously $\min M' \geq \ell_0$, therefore $mn < \min M' + 1$ and $\min M' \geq n$.

Finally for $s = \pi(\min M')$ we have

$$U = C_{\pi|M'} \setminus \bigcup_{i=1}^{m} C_{\pi_i} \in \Gamma(s).$$

So U is a required neighbourhood. □

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Chair of Algebra and Topology, Mathematical Department, Udmurt State University, Izhevsk, Russia

E-mail address: `gryzlov@uni.udm.ru`

E-mail address: `vporoshok@gmail.com`

E-mail address: `rpa4@bk.ru`