Subsemigroups of $\beta S$ Containing the Idempotents

by

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CONTAINING THE IDEMPOTENTS

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ABSTRACT. Let $S$ be a discrete semigroup and let $P(S)$ be the set of points $p$ in the Stone-Cech compactification, $\beta S$, of $S$ with the property that every neighborhood of $p$ contains arbitrarily large finite sum sets or finite product sets, (depending on whether the operation in $S$ is denoted by $+$ or $\cdot$). Then $P(S)$ contains all of the idempotents of $\beta S$, where the operation on $\beta S$ extends that on $S$ making $\beta S$ into a right topological semigroup with $S$ contained in its topological center. If $S$ is commutative, then $P(S)$ is a compact subsemigroup of $\beta S$. Responding to a question of Vitaly Bergelson, we show that if $S$ is any semigroup which can be embedded in a compact topological group, then $P(S)$ is not the smallest closed semigroup containing the idempotents of $\beta S$ and the closure of the semigroup generated by the idempotents of $\beta S$ is not a semigroup.

1. INTRODUCTION

In 1933 Richard Rado published [8] his remarkable theorem characterizing those finite matrices with rational coefficients which are kernel partition regular over the set $\mathbb{N}$ of positive integers. (A $u \times v$ matrix $A$ is kernel partition regular over $\mathbb{N}$ if and only if whenever $\mathbb{N}$ is partitioned into finitely many classes, there will exist $\vec{x} \in \mathbb{N}^v$ with all of its entries in one class such that $A\vec{x} = \vec{0}$.)

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As an easy consequence, one sees that the matrix \( \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \) is kernel partition regular over \( \mathbb{N} \). That is, whenever \( \mathbb{N} \) is partitioned into finitely many cells, there will be in one cell some \( x, y, \) and \( z \) with \( x + y = z \). This result is Schur’s Theorem [10]. More generally, it is an easy consequence of Rado’s Theorem that whenever \( r \in \mathbb{N} \) and \( \mathbb{N} \) is divided into finitely many cells, there will exist a finite sequence \( \langle x_t \rangle_{t=1}^r \) in \( \mathbb{N} \) such that \( \mathcal{F}S(\langle x_t \rangle_{t=1}^r) \) is contained in one cell, where \( \mathcal{F}S(\langle x_t \rangle_{t=1}^r) = \{ \sum_{t \in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, r\} \} \). (See [4, Corollary 2.4] for the details of how this follows easily from Rado’s Theorem.) Much later Jon Sanders [9] and Jon Folkman (unpublished) independently derived this same result.

In [3] an infinite version of this result was obtained. That is, whenever \( \mathbb{N} \) is divided into finitely many cells, there will exist a sequence \( \langle x_t \rangle_{t=1}^\infty \) in \( \mathbb{N} \) such that \( \mathcal{F}S(\langle x_t \rangle_{t=1}^\infty) \) is contained in one cell, where \( \mathcal{F}S(\langle x_t \rangle_{t=1}^\infty) = \{ \sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \} \). (Here \( \mathcal{P}_f(X) \) is the set of finite nonempty subsets of \( X \).) The proof given in [3] was excruciatingly complicated. There is a much simpler proof due to Fred Galvin and Steven Glazer, not published by either of them. See the Notes to [6, Chapter 5] for the history of the discovery of this proof.

**Theorem 1.1.** Let \( A \subseteq \mathbb{N} \). There exists a sequence \( \langle x_t \rangle_{t=1}^\infty \) in \( \mathbb{N} \) with \( \mathcal{F}S(\langle x_t \rangle_{t=1}^\infty) \subseteq A \) if and only if there exists an idempotent \( p \) in \((\beta \mathbb{N}, +)\) such that \( A \in p \).

**Proof.** [6, Theorem 5.12]. \( \square \)

In fact Theorem 1.1 holds more generally. Given any semigroup \((S, \cdot)\), not necessarily commutative, and given a sequence \( \langle x_t \rangle_{t=1}^\infty \) in \( S \), one defines \( \mathcal{F}P(\langle x_t \rangle_{t=1}^\infty) = \{ \prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \} \), where the product \( \prod_{t \in F} x_t \) is taken in increasing order of indices. One then has that for any \( A \subseteq S \) there exists a sequence \( \langle x_t \rangle_{t=1}^\infty \) in \( S \) with \( \mathcal{F}P(\langle x_t \rangle_{t=1}^\infty) \subseteq A \) if and only if there exists an idempotent \( p \) in \((\beta S, \cdot)\) such that \( A \in p \).

Given a discrete space \( X \), we are taking the points of \( \beta X \) to be the ultrafilters on \( X \), identifying the principal ultrafilters with the points of \( X \) and thereby pretending that \( X \subseteq \beta X \). We let \( X^* = \beta X \setminus X \). Given \( A \subseteq X \), \( \overline{A} = \text{cl}_{\beta X} A = \{ p \in \beta S : A \in p \} \). If \((S, \cdot)\) is a discrete semigroup, the operation extends to \( \beta S \) making \((\beta S, \cdot)\) a right topological semigroup (meaning that for each \( p \in \beta S \), the
function \( \rho_p : \beta S \to \beta S \) defined by \( \rho_p(q) = q \cdot p \) is continuous) with \( S \) contained in its topological center (meaning that for each \( x \in S \), the function \( \lambda_x : \beta S \to \beta S \) defined by \( \lambda_x(q) = x \cdot q \) is continuous). Given \( p, q \in \beta S \) and \( A \subseteq S \), one has that \( A \in p \cdot q \) if and only if \( \{ x \in S : x^{-1}A \in q \} \in p \), where \( x^{-1}A = \{ y \in S : x \cdot y \in A \} \). If the operation in \( S \) is denoted by +, we have that \( A \in p + q \) if and only if \( \{ x \in S : -x + A \in q \} \in p \), where \( -x + A = \{ y \in S : x + y \in A \} \). It is a fundamental fact, due originally to R. Ellis [1], that any compact Hausdorff right topological semigroup has an idempotent. See [6] for an elementary introduction to the structure of \( \beta S \).

Recently Vitaly Bergelson asked whether there is some nice algebraic description of the set of ultrafilters on \( \mathbb{N} \), every member of which contains arbitrarily large finite sums sets. This would be the set \( P(\mathbb{N}) \) defined below. (We state the definition multiplicatively because we will be dealing with these sets in a quite general context.)

**Definition 1.2.** Let \((S, \cdot)\) be a semigroup.

(a) For each \( r \in \mathbb{N} \),

\[
P_r(S) = \{ p \in S^* : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^r)(FP(\langle x_t \rangle_{t=1}^r) \subseteq A) \}.
\]

(b) \( P(S) = \bigcap_{r=1}^{\infty} P_r(S) \).

If \( S \) is commutative, it is easy to see that \( P_r(S) \) is a compact subsemigroup of \( \beta S \). (Given \( r \in \mathbb{N} \), \( p, q \in P_r(S) \), and \( A \in p \cdot q \), one has that \( B = \{ x \in S : x^{-1}A \in q \} \in p \) so pick \( \langle x_t \rangle_{t=1}^r \) with \( FP(\langle x_t \rangle_{t=1}^r) \subseteq B \). Then \( C = \bigcap \{ z^{-1}A : z \in FP(\langle x_t \rangle_{t=1}^r) \} \subseteq q \) so pick \( \langle y_t \rangle_{t=1}^r \) with \( FP(\langle y_t \rangle_{t=1}^r) \subseteq C \). Then \( FP(\langle x_t \cdot y_t \rangle_{t=1}^r) \subseteq A \).) By [5, Theorem 3.9] (using a result of Nešetřil and Rödl [7]), for each \( r > 1 \), \( P_{r+1}(\mathbb{N}, +) \) is a proper subset of \( P_r(\mathbb{N}, +) \). Further, it is an immediate consequence of Theorem 1.1 that all idempotents of \( \beta \mathbb{N} \) are in \( P(\mathbb{N}, +) \). Thus, a tempting answer to Bergelson’s question would be that \( P(\mathbb{N}, +) \) is the smallest compact subsemigroup of \( (\beta \mathbb{N}, +) \) containing the idempotents.

However, it was shown in [5] that the closure of the semigroup generated by the idempotents of \( (\beta \mathbb{N}, +) \) is not a semigroup and that there is a compact subsemigroup of \( (\beta \mathbb{N}, +) \) (denoted there by \( M \)) which lies strictly between the smallest subsemigroup of \( (\beta \mathbb{N}, +) \) containing the idempotents and \( P(\mathbb{N}, +) \). In Section 2 of this paper we extend these results to semigroups which can be algebraically embedded in compact topological groups.
In Section 3 we restrict our attention to $P(N, +)$, noting that as a consequence of the following result, $P(N, +)$ is an ideal of $(\beta N, \cdot)$.

**Theorem 1.3.** Let $r \in N$. Then $P_r(N, +)$ is an ideal of $(\beta N, \cdot)$.

**Proof.** Let $p \in P_r(N, +)$ and let $q \in \beta N$. To see that $p \cdot q \in P_r(N, +)$, let $A \in p \cdot q$. Pick $\langle x_t \rangle_{t=1}^r$ such that $FS(\langle x_t \rangle_{t=1}^r) \subseteq \{y \in N : y^{-1}A \in q\}$. Then $B = \bigcap\{y^{-1}A : y \in FS(\langle x_t \rangle_{t=1}^r)\} \in q$ so pick $a \in B$. Then $FS(\langle a \cdot x_t \rangle_{t=1}^r) \subseteq A$.

To see that $q \cdot p \in P_r(N, +)$, let $A \in q \cdot p$. Pick $a \in N$ such that $a^{-1}A \in p$ and pick $\langle x_t \rangle_{t=1}^r$ such that $FS(\langle x_t \rangle_{t=1}^r) \subseteq a^{-1}A$. Then $FS(\langle a \cdot x_t \rangle_{t=1}^r) \subseteq A$. $\Box$

Another tempting answer to Bergelson’s question then becomes that $P(N, +)$ is the smallest compact subset of $\beta N$ which is both a subsemigroup of $(\beta N, +)$ and an ideal of $(\beta N, \cdot)$. We show in Section 3 that this is not the case.

All hypothesized topological spaces are Hausdorff.

### 2. Semigroups embeddable in compact topological groups

We show in this section that if $S$ is any semigroup which can be embedded in a compact topological group, then the closure of the semigroup generated by the idempotents of $S^*$ is not a semigroup. (As is well known, such semigroups include all free semigroups and all commutative cancellative semigroups.) We also show, under the same assumption on $S$, that there is an element of $P(S)$ which is not a member of the smallest compact subsemigroup of $\beta S$ containing the idempotents of $S^*$. (This result is less interesting in the case that $S$ is not commutative, since then it is unlikely that $P(S)$ will be a semigroup.)

The following lemma is, as we are fond of saying, well known by those who know it well.

**Lemma 2.1.** Let $(S, \cdot)$ be a countably infinite semigroup. If $S$ can be algebraically embedded in a compact topological group, then $S$ can be algebraically embedded in a compact metrizable topological group.
Proof. Let $G$ be a compact topological group with identity $1$ and let $\varphi : S \to G$ be an injective homomorphism. Let

$$H = \{ \varphi(s)\varphi(t)^{-1} : s, t \in S \text{ and } s \neq t \}.$$  

For $a \in H$ pick by [2, Theorem 22.14] a compact metrizable topological group $C_a$ and a continuous homomorphism $h_a : G \to C_a$ such that $h_a(a) \neq h_a(1)$. Let $C = \times_{a \in H} C_a$ and define $\psi : S \to C$ by $\psi(s)(a) = h_a(\varphi(s))$ for each $a \in H$. Given $s \neq t$ in $S$, if $a = \varphi(s)\varphi(t)^{-1}$, then $\psi(s)(a) \neq \psi(t)(a)$ so $\psi$ is injective.

□

The Lemma 2.3 will be used in the proofs of both of the theorems of this section. If $n \in \mathbb{N}$, supp$(n)$ is the subset of $\omega$ determined by $n = \sum_{t \in \text{supp}(n)} 2^t$, where $\omega = \mathbb{N} \cup \{0\}$.

**Definition 2.2.**

(a) $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(\mathbb{N}2^n)$.

(b) Let $X$ be a subset of a semigroup. A function $\psi : \omega \to X$ will be called an $\mathbb{H}$-map if it is bijective and if $\psi(m + n) = \psi(m)\psi(n)$ whenever $m, n \in \mathbb{N}$ satisfy $\max \text{supp}(m) + 1 < \min \text{supp}(n)$.

Note that by [6, Lemma 6.6], $\mathbb{H}$ contains all of the idempotents of $(\beta\mathbb{N}, +)$.

**Lemma 2.3.** Let $S$ be a countable semigroup which can be embedded in a compact topological group. Then there exist a countable group $G$ containing $S$, an $\mathbb{H}$-map $\psi : \omega \to G$, and a subsemigroup $V$ of $G^*$ which contains all of the idempotents of $G^*$ such that $\tilde{\psi}|_{\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto $V$. Further, there is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in $S$ such that for each $n$, $\max \text{supp} \psi^{-1}(s_n) + 1 < \min \text{supp} \psi^{-1}(s_{n+1})$.

**Proof.** By Lemma 2.1 there exist a compact metrizable topological group $C$ with identity $1$ and an injective homomorphism $\varphi : S \to C$. Let $G$ be the subgroup of $C$ generated by $\varphi[S]$ and let $\beta G_d$ be the Stone-Čech compactification of $G$ with the discrete topology. We may assume in fact that $S \subseteq G$. Let $\iota : G \to C$ be the inclusion map and let $\tilde{\iota} : \beta G_d \to C$ be its continuous extension. Let $V = G^* \cap \tilde{\iota}^{-1}\{1\}$. By [6, Theorem 7.28] $V$ is a subsemigroup of $G^*$ which contains all of the idempotents of $G^*$ and there is an $\mathbb{H}$-map $\psi : \omega \to G$ such that $\tilde{\psi}|_{\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto $V$.
Now pick an idempotent $q \in S^*$. (By [6, Theorem 4.36] $S^*$ is a subsemigroup of $\beta S$ so has an idempotent.) We choose the sequence $(s_n)_{n=1}^\infty$ inductively, letting $s_1$ be any element of $S$. Let $n \in \mathbb{N}$ and assume that $s_1, s_2, \ldots, s_n$ have been chosen. Let $k = \max \operatorname{supp} \psi^{-1}(s_n) + 2$. Now $q \in V$ so $\psi^{-1}(q)$ is an idempotent in $\mathbb{H}$ and thus $\mathbb{N}2^k \in \psi^{-1}(q)$. By [6, Lemma 3.30] $\psi[\mathbb{N}2^k] \in q$ so pick $s_{n+1} \in \psi[\mathbb{N}2^k]$. □

Note that the idempotents $p_n$ hypothesized in the next lemma exist by [6, Lemma 5.11].

**Lemma 2.4.** Let $(x_t)_{t=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all $t$, $\max \operatorname{supp}(x_t) < \min \operatorname{supp}(x_{t+1})$. Let $\{E_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets and for each $n$ let $p_n$ be an idempotent in $\beta \mathbb{N}$ such that for each $m \in \mathbb{N}$,

$$\{\sum_{t \in F} x_t : F \in \mathcal{P}_f(E_n) \text{ and } \min F > m\} \in p_n.$$ 

Let $p$ be a cluster point in $\beta \mathbb{N}$ of the sequence $(p_n)_{n=1}^\infty$ and let $A = \{\sum_{t \in F} x_t + \sum_{t \in G} x_t : F \in \mathcal{P}_f(E_1) \text{ and } (\exists n)(\max F < n < \min G \text{ and } G \in \mathcal{P}_f(E_n))\}$. Then $A \in p_1 + p$ and there do not exist $r \in \mathbb{H}$ and an idempotent $q$ such that $A \in r + q$.

**Proof.** To see that $A \in p_1 + p$ we show that

$$FS((x_t)_{t \in E_1}) \subseteq \{a \in \mathbb{N} : -a + A \in p\}.$$ 

So let $F \in \mathcal{P}_f(E_1)$, let $a = \sum_{t \in F} x_t$, and let

$$B = \{\sum_{t \in G} x_t : (\exists n)(\max F < n < \min G \text{ and } G \in \mathcal{P}_f(E_n))\}.$$ 

Then $B \subseteq -a + A$ so it suffices to show that $B \in p$. Suppose instead $B \notin p$ and pick $n > \max F$ such that $p_n \notin \mathbb{N} \setminus B$. Then $\{\sum_{t \in G} x_t : G \in \mathcal{P}_f(E_n) \text{ and } \min G > n\}$ is an element of $p_n$ which is contained in $B$, a contradiction.

Now suppose that we have $r \in \mathbb{H}$ and an idempotent $q$ such that $A \in r + q$. Let $X = FS((x_t)_{t \in E_1})$. We claim first that $X \in q$, so suppose instead that $X \notin q$. Pick $a \in \mathbb{N}$ such that $-a + A \in q$ and pick $k \in \mathbb{N}$ such that $\max \operatorname{supp}(a) < \min \operatorname{supp}(x_k)$ and let $m = \max \operatorname{supp}(x_k) + 1$. Pick $b \in (-a + A) \cap \mathbb{N}2^m \setminus X$. Then $a + b = \sum_{t \in F} x_t + \sum_{t \in G} x_t$, where $F \in \mathcal{P}_f(E_1)$ and there is some $n$ with $\max F < n < \min G$ and $G \in \mathcal{P}_f(E_n)$. Let $H = \{t \in F \cup G : t < k\}$ and let $K = \{t \in F \cup G : t > k\}$. Then since $\operatorname{supp}(x_k) \cap \operatorname{supp}(a + b) = \emptyset$, ...
we have $H \cup K = F \cup G$. Also, $\max \text{supp}(a) < \min \text{supp}(\sum_{t \in K} x_t)$ and $\max \text{supp}(\sum_{t \in H} x_t) < \min \text{supp}(b)$ so $a = \sum_{t \in H} x_t$ and $b = \sum_{t \in K} x_t$, so $b \in X$, a contradiction.

Define $g : X \to \mathbb{N}$ by $g(\sum_{t \in F} x_t) = n$ if and only if $\max F \in E_n$. We next claim that there is some $n \in \mathbb{N}$ such that

$$\{y \in X : g(y) \leq n\} \in q.$$ 

So suppose instead that for all $n \in \mathbb{N}$, $\{y \in X : g(y) > n\} \in q$. Pick $a \in \mathbb{N}$ such that $-a + A \in q$. Let $l = \max \text{supp}(a)$. Now

$$\{b \in (-a + A) : -b + (-a + A) \in q\} \in q,$$

so pick $b \in (-a + A) \cap X \cap \mathbb{N}^{2^{l+1}}$ such that $-b + (-a + A) \in q$. Pick $F \in \mathcal{P}_f(E_1)$ and $n \in \mathbb{N}$ such that $\max F < n < \min G$, $G \in \mathcal{P}_f(E_n)$, and $a + b = \sum_{t \in F} x_t + \sum_{t \in G} x_t$. Then $g(b) = n$. Let $k = \max \text{supp}(b)$ and pick

$$b' \in (-b + (-a + A)) \cap X \cap \mathbb{N}^{2^{k+1}} \cap \{y \in X : g(y) > n\}.$$

Then $a + b + b' \in A$ so $a + b + b' = \sum_{t \in F'} x_t + \sum_{t \in G'} x_t$ for some $F' \in \mathcal{P}_f(E_1)$ and some $G' \in \mathcal{P}_f(E_m)$ where $\max F' < m < \min G'$. Then $m = g(b')$, so $m > n$. But then $a + b + b' = \sum_{t \in H} x_t$ where $H \cap E_1 \neq \emptyset$, $H \cap E_n \neq \emptyset$, and $H \cap E_m \neq \emptyset$, a contradiction. Thus we do have some $n \in \mathbb{N}$ such that $\{y \in X : g(y) \leq n\} \in q$.

Now let $k = \max \text{supp}(x_n)$ and pick $a \in \mathbb{N}^{2^{k+1}}$ such that $-a + A \in q$ (using the fact here that $r \in \mathbb{H}$). Let $l = \max \text{supp}(a)$ and pick $b \in (-a + A) \cap \mathbb{N}^{2^{l+1}} \cap \{y \in X : g(y) \leq n\}$. Pick $F \in \mathcal{P}_f(E_1)$ and $m$ and $G$ such that $G \in \mathcal{P}_f(E_m)$ and $\max F < m < \min G$. Then $g(b) = m$ so $m \leq n$. But $\min \text{supp}(a + b) > k$ so $\max F > n$ so $m > n$, a contradiction.$\Box$

Recall that a semigroup $(S, \cdot)$ is weakly cancellative provided that for all $x, y \in S$, $\{s \in S : x \cdot s = y \text{ or } s \cdot x = y\}$ is finite.

**Lemma 2.5.** Let $(S, \cdot)$ be an infinite weakly cancellative semigroup. Then there is a countable subsemigroup $T$ of $S$ such that if $q, r \in \beta S$, $q = q \cdot q$, and $r \cdot q \in \overline{T}$, then $r \in \overline{T}$ and $q \in \overline{T}$. Furthermore, if $A$ is the subsemigroup of $\overline{T}$ generated by the idempotents of $T^*$ and $B$ is the subsemigroup of $\beta S$ generated by the idempotents of $S^*$, then $\text{cl}A = \overline{T} \cap \text{cl}B$. 


Lemma 2.3 a countable group

Proof. Let \( C_1 \) be an arbitrary countable subsemigroup of \( S \). Given \( n \in \mathbb{N} \) and \( C_n \), let

\[
D_n = C_n \cup \{ s \in S : (\exists x \in C_n)(x \cdot s \in C_n \text{ or } s \cdot x \in C_n) \}
\]

and let \( C_{n+1} \) be the semigroup generated by \( D_n \). Let \( T = \bigcup_{n=1}^{\infty} C_n \). Trivially \( T \) is a countable subsemigroup of \( S \). Notice also that if \( x \in T \), \( s \in S \), and either \( xs \in T \) or \( sx \in T \), then \( s \in T \).

Now assume that \( q, r \in \beta S \), \( q = q \cdot q \), and \( r \cdot q \in T \). Then \( T \cap r \cap q = r \cap q \cdot q \cdot q \) so \( \{ x \in S : x^{-1}T \in q \} \subseteq r \cdot q \). Pick \( x \in T \) such that \( x^{-1}T \in q \). Then \( x^{-1}T \subseteq T \) so \( T \in q \).

Now \( \{ s \in S : s^{-1}T \in q \} \subseteq r \). We claim that

\[
\{ s \in S : s^{-1}T \in q \} \subseteq T
\]

so that \( T \in r \). Let \( s \in S \) such that \( s^{-1}T \in q \). Pick \( x \in s^{-1}T \cap T \). Then \( sx \in T \) so \( s \in T \).

One easily shows by induction on \( k \) that if \( k \in \mathbb{N} \) and \( r_1, r_2, \ldots, r_k \) are idempotents in \( \beta S \) and \( r_1 \cdot r_2 \cdots r_k \in T \) then \( \{ r_1, r_2, \ldots, r_k \} \subseteq T \).

Trivially \( c\ell A \subseteq T \cap c\ell B \). For the reverse inclusion, let \( p \in T \cap c\ell B \) and let \( C \in p \). Now \( C \cap T \in p \) so \( C \cap T \cap B \neq \emptyset \) so pick \( k \in \mathbb{N} \) and idempotents \( r_1, r_2, \ldots, r_k \) in \( S^* \) such that \( r_1 \cdot r_2 \cdots r_k \in C \cap T \). Then \( \{ r_1, r_2, \ldots, r_k \} \subseteq T \) so \( r_1 \cdot r_2 \cdots r_k \in C \cap A \).

We are now ready to fulfill the first of our objectives of this section.

Theorem 2.6. Let \( S \) be a semigroup which is embeddable in a compact topological group and let \( B \) be the subsemigroup of \( \beta S \) generated by the idempotents of \( S^* \). Then \( c\ell B \) is not a semigroup. In fact, there exist an idempotent \( q_1 \) of \( S^* \) and a point \( q \) in the closure of the set of idempotents of \( S^* \) such that \( q_1 \cdot q \notin c\ell B \).

Proof. By Lemma 2.5 we may assume that \( S \) is countable. Pick by Lemma 2.3 a countable group \( G \) containing \( S \), an \( H \)-map \( \psi : \omega \to G \), and a subsemigroup \( V \) of \( G^* \) which contains all of the idempotents of \( G^* \) such that \( \overline{\psi} \) is an isomorphism from \( H \) onto \( V \). Also pick a sequence \( \langle s_n \rangle_{n=1}^{\infty} \) in \( S \) such that for each \( n \), \( \max \supp \psi^{-1}(s_n) + 1 < \min \supp \psi^{-1}(s_{n+1}) \). For each \( n \), let \( x_n = \psi^{-1}(s_n) \).

Let \( \{ E_n : n \in \mathbb{N} \} \) be a partition of \( \mathbb{N} \) into infinite sets and for each \( n \) let \( p_n \) be an idempotent in \( \beta \mathbb{N} \) such that for each \( m \in \mathbb{N} \),

\[
\{ \sum_{t \in F} x_t : F \in \mathcal{P}_f(E_n) \text{ and } \min F > m \} \in p_n.
\]
Let $p$ be a cluster point in $\beta N$ of the sequence $\langle p_n \rangle_{n=1}^\infty$ and pick by Lemma 2.4 some $A \in p_1 + p$ such that there do not exist $r \in \mathbb{H}$ and an idempotent $q \in N^*$ such that $A \in r + q$.

Let $q_1 = \tilde{\psi}(p_1)$ and let $q = \tilde{\psi}(p)$. Then $q_1$ is an idempotent of $G^*$ and since $p_1 \in \epsilon\{x_n : n \in \mathbb{N}\}$, $q_1 \in \epsilon\{s_n : n \in \mathbb{N}\}$ so $q_1$ is an idempotent of $S^*$. Similarly, each $\tilde{\psi}(p_n) \in S^*$ so $q \in S^*$ and $q \in \epsilon\{\tilde{\psi}(p_n) : n \in \mathbb{N}\}$.

Now $A \in p_1 + p$ so $\psi[A] = \tilde{\psi}(p_1 + p) = q_1 \cdot q$. Suppose $\overline{\psi[A]} \cap B \neq \emptyset$ and pick $k \in \mathbb{N}$ and idempotents $r_1, r_2, \ldots, r_k$ in $S^*$ such that $\psi[A] \in r_1 \cdot r_2 \cdot \ldots \cdot r_k$. (We may presume that $k \geq 2$, since $r_1 = r_1 \cdot r_1$.) Then $\tilde{\psi}^{-1}(r_1 \cdot r_2 \cdot \ldots \cdot r_k) \notin \mathbb{H}$ and $\tilde{\psi}^{-1}(r_k)$ is an idempotent of $N^*$ and $A \notin \tilde{\psi}^{-1}(r_1 \cdot r_2 \cdot \ldots \cdot r_k) + \tilde{\psi}^{-1}(r_k)$, a contradiction. \hfill \Box

We now turn our attention to showing that under the same hypotheses $P(S)$ is not the smallest compact subsemigroup of $\beta S$ containing the idempotents of $S^*$.

**Lemma 2.7.** Let $S$ and $T$ be discrete semigroups, let $h : S \rightarrow \beta T$ be a homomorphism and let $\tilde{h} : \beta S \rightarrow \beta T$ denote the continuous extension of $h$. Then $\tilde{h}(P(S)) \subseteq P(T)$.

**Proof.** Let $x \in P(S)$, let $B \in \tilde{h}(x)$ and let $n \in \mathbb{N}$ with $n \geq 3$. Pick $C \in x$ such that $\tilde{h}(C) \subseteq B$ and pick $\langle a_t \rangle_{t=1}^n$ such that $FP(\langle a_t \rangle_{t=1}^n) \subseteq C$. We shall construct inductively $\langle b_t \rangle_{t=1}^n$ such that $FP(\langle b_t \rangle_{t=1}^n) \subseteq B$.

Given $z \in FP(\langle h(a_t) \rangle_{t=2}^n)$, we have $h(a_1)z \in \overline{B}$ so pick $D_z \in h(a_1)$ such that $D_z \subseteq \overline{B}$. Also $B \in h(a_1)$ so pick

$$b_1 \in B \cap \bigcap\{D_z : z \in FP(\langle h(a_t) \rangle_{t=2}^n)\}.$$  

Now let $m \in \{1, 2, \ldots, n-2\}$ and assume we have chosen $\langle b_t \rangle_{t=1}^m$ such that for each $c \in FP(\langle b_t \rangle_{t=1}^m)$ and $z \in FP(\langle h(a_t) \rangle_{t=m+1}^n)$, $cz \in \overline{B}$. Given $c \in FP(\langle b_t \rangle_{t=1}^m)$ and $z \in FP(\langle h(a_t) \rangle_{t=m+2}^n)$ one has $h(a_{m+1})z \in \overline{B}$, $ch(a_{m+1}) \in \overline{B}$, and $ch(a_{m+1})z \in \overline{B}$. Since $\lambda_c$ and $\rho_z$ are continuous, we may pick $D_z$, $E_c$, and $F_{c,z}$ in $h(a_{m+1})$ such that $D_z \subseteq \overline{B}$, $cE_c \subseteq \overline{B}$, and $cF_{c,z} \subseteq \overline{B}$. Pick

$$b_{m+1} \in B \cap \bigcap\{D_z : z \in FP(\langle h(a_t) \rangle_{t=m+1}^n)\}$$

$$\cap \bigcap\{E_c : c \in FP(\langle b_t \rangle_{t=1}^m)\}$$

$$\cap \bigcap\{F_{c,z} : c \in FP(\langle b_t \rangle_{t=1}^m) \text{ and } z \in FP(\langle h(a_t) \rangle_{t=m+1}^n)\}.$$  

Having chosen $\langle b_t \rangle_{t=1}^m$, pick for each $c \in FP(\langle b_t \rangle_{t=1}^m)$, $E_c \in h(a_n)$ such that $cE_c \subseteq \overline{B}$ and pick $b_n \in B \cap \{E_c : c \in FP(\langle b_t \rangle_{t=1}^m)\}$. \hfill \Box
Recall that if \( q \in \beta \mathbb{N}, \langle x_n \rangle_{n=1}^{\infty} \) is a sequence in a Hausdorff topological space \( X \), and \( y \in X \), then \( y = q\lim_{n \in \mathbb{N}} x_n \) if and only if whenever \( U \) is a neighborhood of \( y \) in \( X \), \( \{ n \in \mathbb{N} : x_n \in U \} \in q \).

**Lemma 2.9.** Let \((S, \cdot)\) be a semigroup, let \( p \in \beta S \), let \( q \in P(\mathbb{N}, +) \), and let \( r = q\lim_{n \in \mathbb{N}} p^n \). Then \( r \in P(S) \).

**Proof.** This follows immediately from Lemma 2.7 and the observation that the map \( n \mapsto p^n \) is a homomorphism from \((\mathbb{N}, +)\) into \( \beta S \). \( \square \)

Given \( v \in \mathbb{N}^* \) and \( n \in \mathbb{N} \), we write \( n \ast v \) for the sum of \( v \) with itself \( n \) times. (The notation \( n \cdot v \) represents the operation in the semigroup \((\beta \mathbb{N}, \cdot)\), and \( n \ast v \) need not equal \( n \cdot v \). For example, if \( v \) is an idempotent, then \( 2 \ast v = v \) and \( 2 \cdot v \neq v \).)

**Lemma 2.9.** Let \( \langle x_n \rangle_{n=1}^{\infty} \) be a sequence in \( \mathbb{N} \) such that for each \( n \in \mathbb{N} \), \( \max \supp(x_n) < \min \supp(x_{n+1}) \), let \( q \in P(\mathbb{N}) \), let
\[
v \in \{ x_n : n \in \mathbb{N} \}^*,
\]
and let \( p = q\lim_{n \in \mathbb{N}} n \ast v \). Then
\[
p \in \mathbb{H} \cap P(\mathbb{N}) \setminus \text{cl} \bigcup \{ \beta \mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e \}.
\]

**Proof.** By Lemma 2.8 \( p \in P(\mathbb{N}) \). One shows easily by induction on \( n \) that for \( n, k \in \mathbb{N} \), \( \{ \sum_{t \in F} x_t : |F| = n \text{ and } \min F > k \} \subset n \ast v \). In particular, each \( n \ast v \in \mathbb{H} \) and so \( p \in \mathbb{H} \). Let
\[
A = \{ \sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F > |F| \}.
\]
Then given any \( n \in \mathbb{N} \), \( \{ \sum_{t \in F} x_t : |F| = n \text{ and } \min F > n \} \subset A \) so \( A \in p \). We claim that \( A \cap \bigcup \{ \beta \mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e \} = \emptyset \).

Suppose instead we have some \( e = e + e \) such that \( A \cap (\beta \mathbb{N} + e) \neq \emptyset \) and pick \( y \in \mathbb{N} \) such that \( -y + A \in e \). Pick \( l \in \mathbb{N} \) such that \( \min \supp(x_l) > \max \supp(y) \). We claim that for each \( m \geq l \), if \( k = \max \supp(x_m) \), then \( y \in FS(\langle x_t \rangle_{t=m+1}^{\infty}) \subset FS(\langle x_t \rangle_{t=m+1}^{\infty}) \setminus \mathbb{N}^{2k+1} \). So let \( z \in (-y + A) \cap \mathbb{N}^{2k+1} \). Then \( y + z \in A \) so pick \( F \in \mathcal{P}_f(\mathbb{N}) \) such that \( y + z = \sum_{t \in F} x_t \). Let \( H = \{ t \in F : t > m \} \) and let \( K = \{ t \in F : t < m \} \). Now \( \supp(y) \cup \supp(z) = \supp(y + z) = \bigcup_{t \in F} \supp(x_t) = \bigcup_{t \in H} \supp(x_t) \cup \bigcup_{t \in K} \supp(x_t) \). Also \( \max \bigcup_{t \in H} \supp(x_t) \leq \max \supp(x_m) \).
Taking \( l = m \), we have \( y \in FS(\langle x_i \rangle_{i=1}^l) \). Also we have that for all \( m \geq l \), \( FS(\langle x_i \rangle_{i=m+1}^\infty) \in e \).

Given any \( B \in e \), we let \( B^* = \{ z \in B : -z + B \in e \} \). Then by [6, Lemma 4.14], whenever \( z \in B^*, -z + B^* \in e \). Now we choose inductively \( \langle F_j \rangle_{j=1}^l \), with \( \min F_i > l \) and for each \( i \in \{1, 2, \ldots, l-1\} \), \( \max F_i < \min F_{i+1} \), with \( \bigcup_{j=1}^l \bigcup_{t \in F_j} x_t \in (-y + A)^* \). Since \((-y + A)^* \in e \), pick \( F_1 \) with \( \min F_1 > l \) such that \( \bigcup_{t \in F_1} x_t \in (-y + A)^* \). Having chosen \( \langle F_j \rangle_{j=1}^l \), let \( m = \max F_i \) and pick \( z \in \left( -\sum_{j=1}^l \bigcup_{t \in F_j} x_t \right) + (-y + A)^* \bigcap FS(\langle x_i \rangle_{i=m+1}^\infty) \) and pick \( F_{i+1} \) with \( \min F_{i+1} \geq m + 1 \) such that \( z = \sum_{t \in F_{i+1}} x_t \). Now \( y + \sum_{j=1}^l \bigcup_{t \in F_j} x_t \in A \) and \( y = \bigcup_{t \in H} x_t \) for some \( H \) with \( \max H \leq l \) so \( \min(H \cup \bigcup_{j=1}^l F_j) \leq l \) while \( |H \cup \bigcup_{j=1}^l F_j| \geq l + 1 \), a contradiction. \( \square \)

**Theorem 2.10.** Let \( S \) be a semigroup which can be embedded in a compact topological group. Let

\[
L = e \ell \bigcup \{ \beta S \cdot e : e \in S^* \text{ and } e = e \cdot e \}.
\]

Then \( L \) is a left ideal of \( \beta S \) and there exists \( r \in P(S) \setminus L \). (So if \( S \) is commutative, \( L \cap P(S) \) is a compact subsemigroup of \( \beta S \) containing the idempotents of \( S^* \) and properly contained in \( P(S) \).)

**Proof.** By [6, Theorem 2.17], \( L \) is a left ideal of \( \beta S \). We first show that it suffices to assume that \( S \) is countable. To see this, pick by Lemma 2.5 a countable subsemigroup \( T \) of \( S \) such that if \( q, r \in \beta S \), \( q = q \cdot q \) and \( r \cdot q \in T \), then \( r \in T \) and \( q \in T \). Assume that we have some \( r \in P(T) \setminus e \ell \bigcup \{ T \cdot e : e \in T^* \text{ and } e = e \cdot e \} \). Then \( r \in P(S) \).

If \( A \in r \) such that \( \overline{A \cap \bigcup T \cdot e : e \in T^* \text{ and } e = e \cdot e} = \emptyset \), then \( A \cap \bigcup \{ \beta S \cdot e : e \in S^* \text{ and } e = e \cdot e \} = \emptyset \). So we shall assume that \( S \) is countable.

Pick by Lemma 2.3 a countable group \( G \) containing \( S \), an \( \mathbb{H} \)-map \( \psi : \omega \to G \), and a subsemigroup \( V \) of \( G^* \) which contains all of the idempotents of \( G^* \) such that \( \psi|_\mathbb{H} \) is an isomorphism from \( \mathbb{H} \) onto \( V \). Also pick a sequence \( \langle s_n \rangle_{n=1}^\infty \) in \( S \) such that for each \( n \), \( \max \text{supp } \psi^{-1}(s_n) + 1 < \min \text{supp } \psi^{-1}(s_{n+1}) \). For each \( n \), let \( x_n = \psi^{-1}(s_n) \). Let \( q \in P(\mathbb{N}) \), let \( v \in \{ x_n : n \in \mathbb{N} \}^* \), and let \( p = q \lim n \ast v \). Then by Lemma 2.9

\[
p \in \mathbb{H} \setminus P(\mathbb{N}) \setminus e \ell \bigcup \{ \beta \mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e \}.
\]
Let \( r = \tilde{\psi}(p) \) and let \( w = \tilde{\psi}(v) \). Then it is routine to establish that 
\[ r = q \lim_{n \to \infty} w^n \] 
and so, by Lemma 2.8 \( r \in P(S) \).

Now we claim that \( r \notin L \). To see this, pick \( A \in p \) such that 
\[ \overline{A} \cap \bigcup \{ \beta S + e : e \in \mathbb{N}^* \text{ and } e + e = e \} = \emptyset. \]
Then \( \psi[A] \in r \). We claim that 
\[ \overline{\psi[A]} \cap \bigcup \{ \beta S \cdot e : e \in \mathbb{N}^* \text{ and } e = e \cdot e \} = \emptyset. \]
Suppose instead that we have some \( e = e \cdot e \in \mathbb{N}^* \) and \( y \in S \) such that 
\( \psi[A] \in y \cdot e \). \( \overline{\psi^{-1}(e)} \) is an idempotent in \( \mathbb{N}^* \), so it suffices to show that 
\( A \in \psi^{-1}(y) + \overline{\psi^{-1}(e)} \). Let \( u = \psi^{-1}(y) \) and let \( k = \max \text{supp}(u) \).

Since \( y^{-1}\psi[A] \in e \) and consequently \( \psi^{-1}[y^{-1}\psi[A]] \in \psi^{-1}(e) \), it suffices to show that 
\( \mathbb{N}2^{k+2} \cap \psi^{-1}[y^{-1}\psi[A]] \subseteq -\psi^{-1}(y) + A \). So
\begin{align*}
&\text{let } z \in \mathbb{N}2^{k+2} \cap \psi^{-1}[y^{-1}\psi[A]]. \\
&\text{Then } y\psi(z) \in \psi[A] \text{ so } \psi(u + z) = \\
&\psi(u)\psi(z) = y\psi(z) \in \psi[A] \text{ so } u + z \in A \text{ and thus } z \in -\psi^{-1}(y) + A.
\end{align*}
\( \square \)

3. \( \beta \mathbb{N} \)

Recall that we have seen that \( P(\mathbb{N}, +) \), in addition to being a compact subsemigroup of \( (\beta \mathbb{N}, +) \) containing the idempotents, is also a two sided ideal of \( (\beta \mathbb{N}, \cdot) \). We see now that it is not the smallest such.

**Theorem 3.1.** There is a compact subsemigroup of \( (\beta \mathbb{N}, +) \) which contains the idempotents of \( (\beta \mathbb{N}, +) \), is a two sided ideal of \( (\beta \mathbb{N}, \cdot) \), and is properly contained in \( P(\mathbb{N}, +) \).

**Proof.** Choose \( v \in \{2^2 : n \in \mathbb{N}\}^* \) and \( q \in P(\mathbb{N}) \). Let \( p = q \lim_{n \to \infty} n \ast v \). Then by Lemma 2.8, \( p \in P(\mathbb{N}) \).

Let
\[ L = \text{cl} \bigcup \{ \beta \mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e \}. \]

Then by Lemma 2.9, \( p \notin L \).

Define \( f : \mathbb{N} \to \mathbb{R} \) by putting \( f(n) = \log_2(n) \), and let \( \tilde{f} : \beta \mathbb{N} \to u\mathbb{R} \) denote the continuous extension of \( f \), where \( u\mathbb{R} \) denotes the uniform compactification of \( \mathbb{R} \). We observe that \( \mathbb{R} \) can be regarded as a subspace of \( u\mathbb{R} \), because \( \mathbb{R} \) can be embedded in \( u\mathbb{R} \) by a topological isomorphism. Then by [11, Lemma 2.1], \( \tilde{f} \) has the following properties:

(i) \( \tilde{f}(x \cdot y) = \tilde{f}(x) + \tilde{f}(y) \) for every \( x, y \in \beta \mathbb{N} \) and
(ii) \( \tilde{f}(x + y) = \tilde{f}(y) \) for every \( x \in \beta \mathbb{N} \) and every \( y \in \mathbb{N}^* \).
For a subset $S$ of $\mathbb{R}$, $\rho(S)$ will denote $\text{cl}_{u\mathbb{R}}(S) \setminus \mathbb{R}$. Let $X = \rho(\{2^n : n \in \mathbb{N}\})$. We claim that $X \subseteq \text{int}_{\rho(\mathbb{R})}(\rho(\mathbb{R}) \setminus (\rho(\mathbb{R}) + \rho(\mathbb{R})))$.

To see this, put $E = \{2^n : n \in \mathbb{N}\}$ and put $F = \mathbb{R} \setminus ((-1, 1) + E)$. Using the uniform structure on $\mathbb{R}$ defined by the usual metric, it follows from [6, Exercise 21.5.3] that there is a uniformly continuous function $\phi : \mathbb{R} \to [0, 1]$ such that $\phi(E) = \{0\}$ and $\phi(F) = \{1\}$. Let $\tilde{\phi} : u\mathbb{R} \to [0, 1]$ denote the continuous extension of $\phi$. If $W = \tilde{\phi}^{-1}([0, \frac{1}{2}])$, then $W$ is an open neighbourhood of $X$ in $u\mathbb{R}$ and $W \subseteq \text{cl}_{u\mathbb{R}}((-1, 1) + E)$. We shall show that $W \cap (\rho(\mathbb{R}) + \rho(\mathbb{R})) = \emptyset$.

To see this, assume that $\xi, \eta \in \rho(\mathbb{R})$ and that $\xi + \eta \in W$. We can choose $s, t \in \mathbb{R}$ with $|s-t| > 2$ such that $s+\eta$ and $t+\eta$ are both in $W$, because $\{\zeta \in u\mathbb{R} : \zeta + \eta \in W\}$ is a neighbourhood of $\xi$ in $u\mathbb{R}$, and so its intersection with $\mathbb{R}$ is unbounded. (If $B \subseteq \mathbb{R}$ is bounded, then $\text{cl}_{u\mathbb{R}}(B) \subseteq \mathbb{R}$.) Then $-s+W$ and $-t+W$ are both neighbourhoods of $\eta$. However, we claim that $(-s+W) \cap (-t+W) \cap \mathbb{R}$ is bounded. To see this, note that for any $x \in (-s+W) \cap (-t+W) \cap \mathbb{R}$ we may pick $n, m \in \mathbb{N}$ such that $|s+x-2^n| < 1$ and $|t+x-2^n| < 1$. Since $|s-t| > 2$ we have that $n \neq m$. On the other hand, $|2^n - 2^m| = |2^n - x - s + t + x - 2^n - s - t| < 2 + |s-t|$. Thus there are only finitely many pairs $(n, m)$ for which there is some $x$ with $|s+x-2^n| < 1$ and $|t+x-2^n| < 1$. Given any such $(n, m)$ and $x$, $|x| \leq |x+s-2^n| + |s-2^n| < 1 + |s-2^n|$ so $(-s+W) \cap (-t+W) \cap \mathbb{R}$ is bounded as claimed. But this contradicts the assumption that $\eta \in \rho(\mathbb{R})$.

It follows from (ii) above that $J = \tilde{f}^{-1}[\rho(\mathbb{R}) \setminus W]$ is a closed subset of $\mathbb{N}^*$ which is a left ideal of $(\beta\mathbb{N}, +)$. Furthermore, it follows from (i) above that $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq J$, and in particular $J$ is a two sided ideal of $(\mathbb{N}^*, \cdot)$. Let $V$ denote the smallest compact subset of $\beta\mathbb{N}$ which is both a left ideal of $(\beta\mathbb{N}, +)$ and satisfies $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq V$. Then $V \subseteq J$. We claim that $V$ is an ideal of $(\beta\mathbb{N}, \cdot)$. To see this, let $n \in \mathbb{N}$. Then by [6, Theorem 6.54] $nV = Vn$ so it suffices to show that $nV \subseteq V$. To see this, let $W = \{p \in \beta\mathbb{N} : n \cdot p \in V\}$. Then it is easy to verify that $W$ is a compact subset of $\beta\mathbb{N}$ which is both a left ideal of $(\beta\mathbb{N}, +)$ and satisfies $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq W$, and consequently $V \subseteq W$ and therefore $nV \subseteq V$ as required.

We claim that $L \cup V$ is a closed left ideal of $(\beta\mathbb{N}, +)$ and an ideal of $(\beta\mathbb{N}, \cdot)$. It is obviously a closed left ideal of $(\beta\mathbb{N}, +)$. To see that it is an ideal of $(\beta\mathbb{N}, \cdot)$, it is routine to verify that for any $n \in \mathbb{N}$, $n \cdot L = L \cdot n \subseteq L$. Also, for any $x \in \mathbb{N}^*$, $(x \cdot L) \cup (L \cdot x) \subseteq \mathbb{N}^* \cdot \mathbb{N}^* \subseteq V$. SUBSEMIGROUPS OF $\beta S$ 245
We claim that the element \( p \in P(\mathbb{N}) \) defined above is not in \( V \).

To see this, observe that \( \tilde{f}(v) \in \text{cl}_{uR} (E) \) and hence, by property (ii) above, that \( \beta \mathbb{N} + v \subseteq \tilde{f}^{-1}[\text{cl}_{uR} E] \subseteq \tilde{f}^{-1}[W] \). So \( (\beta \mathbb{N} + v) \cap J = \emptyset \) and consequently \( (\beta \mathbb{N} + v) \cap V = \emptyset \). Now let \( r = \lim_{n \in \mathbb{N}} (n - 1) * v \).

Then \( p = r + v \in \beta \mathbb{N} + v \), so \( p \notin V \). We have already noted that \( p \notin L \). Thus \( P(\mathbb{N}) \nsubseteq L \cup V \). \( \square \)

References

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