Extension Theory and the First Uncountable Ordinal Space

by

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Electronically published on December 10, 2009
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ABSTRACT. We shall examine the extension theory of products \( Y = Z \times [0, \Omega) \) where \( Z \) is a compact metrizable space and \( \Omega \) is the first uncountable ordinal. Our main result is that if a CW-complex \( K \) is an absolute extensor for \( Z \), then \( K \) is an absolute extensor for \( Y \). This implies, as a corollary, the classical fact that \( Y \) is normal. We shall also examine the extension theory of pseudo-compact spaces and will prove that if \( X \) is a normal, Hausdorff, pseudo-compact space, and \( K \) is an absolute extensor for \( X \), then it is also an absolute extensor for the Stone-\( \check{C} \)ech compactification of \( X \). From this we will be able to deduce that for the preceding space \( Y \), \( K \) is an absolute extensor for \( \beta(Y) \).

1. Introduction

Let \( X \) and \( K \) be spaces; suppose that for all closed subsets \( A \) of \( X \) and for every map \( f : A \rightarrow K \) there exists a map \( F : X \rightarrow K \) such that \( F|A = f \). Then we write \( X \tau K \) and say either that \( X \) is an absolute co-extensor for \( K \) or \( K \) is an absolute extensor for \( X \). This is the fundamental notion of extension theory (see [1] or [4]) where usually \( K \) is a CW-complex. It then follows that \( X \) is a normal space if and only if \( X \tau \mathbb{R} \).

Let \( \Omega \) designate the first uncountable ordinal. Then \([0, \Omega)\) will denote the set of ordinals less than \( \Omega \) with the order topology,
often called the *first uncountable ordinal space* [6]. An important
tool in the study of \([0, \Omega)\) is its pseudo-compactness. We shall
examine this property in section 2. In Proposition 2.4 we exhibit
the extension-theoretic relation between a pseudo-compactum \(X\)
and its Stone-Čech compactification \(\beta(X)\).

In section 3 we study the extension theory of products \(Z \times [0, \Omega)\)
where \(Z\) is compact and metrizable. Our main theorem, Theorem
3.11, states that if \(Z\) is a compact metrizable space, \(K\) is a CW-
complex, \(Z \tau K\), and \(Y = Z \times [0, \Omega)\), then both \(Y \tau K\)
and \(\beta(Y) \tau K\).

The author wishes to thank Professor Sibe Mardešić for several
important discussions about this subject and Professor Ivan Ivanišić
for his help in the preparation of this paper.

2. Extension theory and pseudo-compacta

A space \(X\) is called *pseudo-compact* if for each map \(f : X \to \mathbb{R}\),
\(f(X)\) is contained in a compact subset of \(\mathbb{R}\), or, equivalently, \(f(X)\)
is a compact subset of \(\mathbb{R}\). A good source of facts about such spaces can
be found in [3]. Here is some information about pseudo-compact
spaces.

**Lemma 2.1.** Let \(X\) be a pseudo-compact space.

(1) If \(Y\) is a compact space, then \(X \times Y\) is pseudo-compact.
(2) If \(X\) is normal and \(A\) is closed in \(X\), then \(A\) is pseudo-
compact.

**Lemma 2.2.** Let \(X\) be a space. The following are equivalent.

(1) \(X\) is pseudo-compact.
(2) For each CW-complex \(K\) and map \(f : X \to K\), \(f(X)\) is
contained in a compact subset of \(K\).
(3) For each CW-complex \(K\) and map \(f : X \to K\), \(f(X)\) is a
compact subset of \(K\).

**Proof:** (1) ⇒ (2). Suppose that \(f(X)\) is not contained in a
compact subset of \(K\). Then there exists a countably infinite closed
discrete subspace \(A\) of \(K\) such that \(A \subset f(X)\). Let \(g : A \to \mathbb{R}\)
be a function such that \(g(A) = \mathbb{N}\). Then \(g\) is a map, and since \(K\)
is normal, there exists a map \(h : K \to \mathbb{R}\) such that \(h|A = g\). Define
\(F = h \circ f : X \to \mathbb{R}\). Then \(F\) is a map of \(X\) to \(\mathbb{R}\). But \(\mathbb{N} \subset F(X)\), so
\(F(X)\) is not contained in a compact subset of \(\mathbb{R}\), a contradiction.
(2) \Rightarrow (3). Let \( L \) be the minimum subcomplex such that \( f(X) \subset L \). Then \( L \) is compact and metrizable. Suppose that \( f(X) \) is not compact; then \( f(X) \) is not closed in \( L \). Let \( p \in L \setminus f(X) \). Since \( L \) is minimal, \( p \) cannot be isolated. It then follows that there is a map \( h : L \setminus \{ p \} \to \mathbb{R} \) such that for all \( n \in \mathbb{N} \), there exists \( x \in L \setminus \{ p \} \) with \( h(x) > n \). Treating \( f : X \to L \setminus \{ p \} \), put \( F = h \circ f : X \to \mathbb{R} \). Then \( F \) is a map of \( X \) to \( \mathbb{R} \). But \( F(X) \) is not contained in a compact subset of \( \mathbb{R} \), a contradiction.

(3) \Rightarrow (1). This follows from the fact that \( \mathbb{R} \) may be given the structure of a CW-complex. \( \square \)

Now we investigate the relation between pseudo-compact spaces and extension theory.

In the proof of Proposition 2.4 we shall use Lemma 2.11 of [7], which we state here for the convenience of the reader.

**Lemma 2.3.** Let \( Y \) be a compact Hausdorff space and \( X \) a dense subset of \( Y \). Then for each closed subset \( A \) of \( Y \) and neighborhood \( G \) of \( A \), there exists a closed neighborhood \( N \) of \( A \) such that \( N \subset G \) and \( N \cap X = N \). \( \square \)

If \( X \) is a Tychonoff space, then \( \beta(X) \) will denote its Stone-Čech compactification. Here is the principal extension-theoretic fact about pseudo-compacta.

**Proposition 2.4.** Let \( X \) be a normal, Hausdorff, pseudo-compact space and \( K \) a CW-complex. Suppose that \( X \tau K \). Then \( \beta(X) \tau K \).

**Proof:** Of course \( X \) is a Tychonoff space. Let \( A \) be a closed subset of \( \beta(X) \) and \( f : A \to K \) a map. Using the fact that \( K \) is an absolute neighborhood extensor for \( \beta(X) \) along with Lemma 2.3, we may as well assume that \( \overline{A \cap X} = A \). Since \( X \tau K \), there is a map \( f_0 : X \to K \) such that \( f_0|A \cap X = f|A \cap X \). By Lemma 2.2, the image of \( f_0 \) lies in a compact subset of \( K \). Hence, there is a map \( F : \beta(X) \to K \) such that \( F|X = f_0|X \). Since \( A \cap X \) is dense in \( A \) and \( F|A \cap X = f_0|A \cap X = f|A \cap X \), then \( F|A = f \), and our proof is complete. \( \square \)

3. Extension theory and \([0, \Omega)\)

It is well known that for each compact metrizable space \( Z \), \( Z \times [0, \Omega) \) is normal. We plan to present a proof of this fact that will lend itself to a generalization into extension theory.
We shall provide a proof of the following known fact.

**Theorem 3.1.** Let $K$ be a CW-complex, $X$ be a compact Hausdorff space with $X \tau K$, and $Y$ be a compact Hausdorff space with $Y \tau S^0$. Then,

$$(Y \times X) \tau K.$$  

Note that for a compact Hausdorff space $Y$, $Y \tau S^0$ is equivalent to $\dim Y \leq 0$.

Before presenting our proof of Theorem 3.1, let us introduce two lemmas. The first is the “tube” lemma (see [2, XI.2.6, p. 228] or [6, 3.26.8, p. 168]).

**Lemma 3.2.** Let $X$ and $Y$ be spaces, $Y$ be compact, $A$ be a subset of $X$, and $U$ be a neighborhood of $A \times Y$ in $X \times Y$. Then there exists a neighborhood $V$ of $A$ in $X$ such that $V \times Y \subset U$. \hfill $\square$

**Lemma 3.3.** Let $X$ be a space such that $X \tau S^0$, $A$ be a closed subset of $X$, and $U$ be a neighborhood of $A$ in $X$. Then there exists an open and closed neighborhood $V$ of $A$ in $X$ such that $V \subset U$.

**Proof:** Let $f : A \cup (X \setminus U) \to S^0$ be the map with $f(A) \subset \{0\}$ and $f(X \setminus U) \subset \{1\}$. Since $X \tau S^0$, there exists a map $F : X \to S^0$ that extends $f$. Let $V = F^{-1}(\{0\})$. It is easy to check that $A \subset V \subset U$. \hfill $\square$

Now we give our proof of Theorem 3.1.

**Proof:** We may as well assume that $Y \neq \emptyset$. Let $A$ be a closed subset of $Y \times X$ and $f : A \to K$ a map. Fix $y \in Y$ and consider the closed subspace $P_y = A \cup \{(y) \times X\} \subset Y \times X$. Since $X \tau K$, there exists a map $f_y : P_y \to K$ such that $f_y|A = f$.

Now $Y \times X$ is compact and Hausdorff; hence, $K$ is an absolute neighborhood extensor for $Y \times X$. So there exists a neighborhood $U_y$ of $P_y$ in $Y \times X$ and a map $G_y : U_y \to K$ extending $f_y$. Using Lemma 3.2 and Lemma 3.3, select an open and closed neighborhood $V_y$ of $y$ in $Y$ such that $V_y \times X \subset U_y$.

There exists a finite subset $\mathcal{F} \subset Y$ such that $\{V_y | y \in \mathcal{F}\}$ covers the compact space $Y$. Write $\mathcal{F} = \{y_1, \ldots, y_n\}$ where $n = \text{card} \mathcal{F}$.

Put $W_1 = V_{y_1}$, and for $1 < k \leq n$, $W_k = V_{y_k} \setminus \bigcup \{V_{y_i} | 1 \leq i < k\}$. Then $\{W_k | 1 \leq k \leq n\}$ is an open and closed cover of $Y$, for each $1 \leq k \leq n$, $W_k \subset V_{y_k}$, and if $1 \leq j < k \leq n$, then $W_j \cap W_k = \emptyset$.  


Define \( F_k = G_{y_k}(W_k \times X) : W_k \times X \to K \). One may now check that
\[
F = \bigcup \{ F_k \mid 1 \leq k \leq n \} : Y \times X \to K
\]
is a map that extends \( f \).

The next is a list of well-known facts about the first uncountable ordinal space.

**Lemma 3.4.** Let \( X = [0, \Omega) \).

(1) \( X \) is a normal Hausdorff space.

(2) Let \( 0 \leq \lambda < \Omega \); then \( [0, \lambda) \) is a compact, 0-dimensional metrizable subspace of \( X \).

(3) Let \( [0, \mu) \) be a closed nonempty subset of \( X \) and \( \sigma \in [0, \mu) \) such that whenever \( \mu, \beta \in P \) and \( \sigma \leq \mu < \beta \), then \( f(\mu) = f(\beta) \).

(4) \( X \) is pseudo-compact. \( \square \)

We now present our main lemma.

**Lemma 3.5.** Let \( Z \) be a nonempty compact metrizable space, \( A \) be a closed subset of \( Y = Z \times [0, \Omega) \) with \( Z \times \{0\} \subset A \), \( f : A \to \mathbb{R} \) be a map, and \( \epsilon > 0 \). For each \( z \in Z \), define \( S_\epsilon(z) = \{ \sigma \in [0, \Omega) \mid \exists \mu, \beta \in [0, \Omega), \sigma \leq \mu < \beta, (z, \mu), (z, \beta) \in A, |f(z, \mu) - f(z, \beta)| \geq \epsilon \} \). Then
\[
\exists \lambda \in [0, \Omega) \text{ such that } \bigcup S_\epsilon(z) \subset [0, \lambda].
\]

**Proof:** Fix \( z \in Z \) and define \( A_z = A \cap \{(z) \times [0, \Omega)\} \). Then \( A_z \) is a nonempty closed subspace of \( \{z\} \times [0, \Omega) \), the latter being a copy of \( [0, \Omega) \). Consider \( f|A_z : A_z \to \mathbb{R} \), and apply Lemma 3.4(3) to this map. Accordingly, there is a first element \( l(z) \in [0, \Omega) \) such that if \( l(z) \leq \mu < \beta \), and \( (z, \mu) \) and \( (z, \beta) \in A \), then \( |f(z, \mu) - f(z, \beta)| \leq \epsilon \).

It then follows that,

(F1) \( S_\epsilon(z) \subset [0, l(z)] \),

(F2) \( l(z) \notin S_\epsilon(z) \), and

(F3) if \( 0 \leq \lambda < l(z) \), then \( \lambda \in S_\epsilon(z) \).

Having defined \( l(z) \in [0, \Omega) \) for each \( z \in Z \) satisfying (F1)–(F3), let us put \( T = \{ l(z) \mid z \in Z, S_\epsilon(z) \neq \emptyset \} \). Suppose we can find \( \alpha \in [0, \Omega) \) so that \( T \subset [0, \alpha] \). Let \( z \in Z \). We claim that \( S_\epsilon(z) \subset [0, \alpha] \). By (F1), \( S_\epsilon(z) \subset [0, l(z)] \). Since \( l(z) \in T \), then \( l(z) \in [0, \alpha] \), so \( S_\epsilon(z) \subset [0, l(z)] \subset [0, \alpha] \). Hence,
(F4) (*) is true if there exists \( \alpha \in [0, \Omega) \) such that \( T \subset [0, \alpha] \).

In case \( T = \emptyset \), then define \( \alpha = 0 \). If \( T \neq \emptyset \) and \( T \) is countable, then put \( \alpha = \sup(T) \). In either case, \( T \subset [0, \alpha] \), so by (F4), (*) is true. Hence, we shall assume that \( T \) is uncountable; we choose an uncountable subset \( Z_0 \subset Z \) so that the function \( l|Z_0 : Z_0 \to T \) is a bijection. To reach a contradiction, suppose that there is no \( \alpha \in [0, \Omega) \) with \( T \subset [0, \alpha] \). This along with (F3) means that

(F5) for all \( \alpha \in [0, \Omega) \), there exists \( z \in Z_0 \) with \( \alpha < l(z) \), and

(F6) for all \( \alpha \in [0, \Omega) \) and \( z \in Z_0 \) with \( \alpha < l(z) \), there are \( \alpha \leq \mu < \beta \) with \( (z, \mu), (z, \beta) \in A \), and \( |f(z, \mu) - f(z, \beta)| \geq \epsilon \).

Employing the well-ordering of \( T \subset [0, \Omega) \), we shall treat \( Z_0 \) as a well-ordered set induced by the bijection \( l|Z_0 : Z_0 \to T \). Let us write \( <_0 \) for the ordering in \( Z_0 \).

Let \( z_0 \) be the first element of \( Z_0 \). Applying (F5) and (F6) with \( \alpha = l(z_0) \), there exist a first element \( a(z_0) \in Z_0 \) such that \( l(z_0) < l(a(z_0)) \), and \( l(z_0) \leq h(z_0) < g(z_0) \), such that \( q(z_0) = (a(z_0), h(z_0)) \in A \), \( r(z_0) = (a(z_0), g(z_0)) \in A \), and \( |f(q(z_0)) - f(r(z_0))| \geq \epsilon \).

We proceed with a transfinite construction. Let \( z \in Z_0 \setminus \{z_0\} \) and suppose that for all \( z \in Z_0 \) with \( z <_0 z \), we have chosen \( a(z) \in Z_0 \) as well as \( l(z) \leq h(z) < g(z) \) such that if \( z <_0 z^* \leq_0 z' <_0 z \), then the following inductive statements are true.

(I1) \( a(z) <_0 a(z^*) \),

(I2) \( g(z) < h(z^*) \) and \( h(z^*) < g(z') \) if \( z^* <_0 z' \),

(I3) \( q(z) = (a(z), h(z)) \in A \), \( r(z) = (a(z), g(z)) \in A \), and

(I4) \( |f(q(z)) - f(r(z))| \geq \epsilon \).

Let \( E = \{z \in Z_0 \mid z <_0 z\} \). Then \( E \) is a countable subset of \( Z_0 \). Put \( B = \{l(a(z)) \mid z \in E\} \), and \( M = \{g(z) \mid z \in E\} \). Each of these sets is a nonempty and countable subset of \([0, \Omega)\). Hence, \( \gamma = \sup(B \cup M) \) exists in \([0, \Omega)\).

Note that \( a(E) \) is a countable subset of \( Z_0 \). Let \( F \) be the subset of \( Z_0 \) consisting of those elements \( u \) with \( a(z) < u \) for all \( z \in E \). Then \( F \) is an uncountable subset of \( Z_0 \). Using this and (F5), there exists \( u \in F \) such that \( \alpha = \max\{l(\gamma), \gamma + 1\} < l(u) \). Define \( a(\gamma) \) to be the first element of \( F \) such that \( \alpha < l(a(\gamma)) \).
Applying (F6) to $\alpha$ and $z = a(\overline{z})$, there are $\alpha \leq h(\overline{z}) < g(\overline{z})$ such that $q(\overline{z}) = (a(\overline{z}), h(\overline{z})) \in A$, $r(\overline{z}) = (a(\overline{z}), g(\overline{z})) \in A$, and $|f(q(\overline{z})) - f(r(\overline{z}))| \geq \varepsilon$.

Let $z \in E$. Then $a(z) < h(\overline{z})$ since $a(\overline{z}) \in F$. Surely $\gamma < h(\overline{z})$. Thus, $h(z) < g(z) < h(\overline{z})$. This ends our inductive construction.

We have defined functions $a : Z_0 \to Z_0$, $h : Z_0 \to [0, \Omega)$, $g : Z_0 \to [0, \Omega)$ such that whenever $z, z^*, z' \in Z_0$ with $z < z^* \leq z'$, statements (I1)–(I4) hold true.

It follows from (I1) that $Q = \{a(z) \mid z \in Z_0\}$ is an uncountable subset of the second countable space $Z_0$. Hence, there exists a point $z_0 \in Z_0$ so that $a(z_0)$ is a limit point of $Q$. So there is a sequence $(y_n)$ in $Q \setminus \{a(z_0)\}$ that converges to $a(z_0)$. Since $Z_0$ is well ordered, so is $Q$, and we may assume that $(y_n)$ is increasing. Let us write $y_n = a(z_n)$ for each $n \in \mathbb{N}$. Because of (I1), $(z_n)$ is increasing.

Applying (I2), one sees that both $(h(z_n))$ and $(g(z_n))$ are increasing sequences in $[0, \Omega)$. So they both converge in $[0, \Omega)$. From (I2) one may conclude that when $m < n < v$ in $\mathbb{N}$, then $g(z_m) < h(z_n) < g(z_v)$. Hence, the sequences $(h(z_n))$ and $(g(z_n))$ have the same limit, say $\rho \in [0, \Omega)$. For each $n \in \mathbb{N}$, let $q_n = (a(z_n), h(z_n))$ and $r_n = (a(z_n), g(z_n))$. By (I3), both $(q_n)$ and $(r_n)$ are sequences in $A$. They converge to $(a(z_0), \rho)$ in $Y$. Since $A$ is closed in $Y$, then $(a(z_0), \rho) \in A$. But $f : A \to \mathbb{R}$ is a map, so each of the sequences $(f(q_n))$ and $(f(r_n))$ converges to the same element $f(a(z_0), \rho)$ of $\mathbb{R}$. This leads to a contradiction, because by (I4), $|f(q_n) - f(r_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Our proof is complete.

Before we can successfully apply Lemma 3.5, we need some additional facts.

**Lemma 3.6.** Let $Z$ be a compact metrizable space. Then the coordinate projection $\pi_Z : Z \times [0, \Omega) \to Z$ is a closed map.

**Proof:** Let $A \subset Z \times [0, \Omega)$ be closed and suppose that $\pi_Z(A)$ is not closed in $Z$. Then there is a sequence $(a_n)$ in $A$ and $z \in Z \setminus \pi_Z(A)$ such that $(\pi_Z(a_n))$ converges to $z$.

For some $0 \leq \lambda < \Omega$, $a_n \in Z \times [0, \lambda]$ for all $n \in \mathbb{N}$. Hence, $(a_n)$ is a sequence in the compact metrizable space $A \cap (Z \times [0, \lambda])$. Passing to a subsequence if necessary, we may assume that $(a_n)$ converges to $a \in A$. Therefore, $\pi_Z(a) = z \in \pi_Z(A)$, a contradiction. \qed
Lemma 3.7. Let $Z$ be a compact metrizable space and $K$ be a CW-complex such that $Z\pi K$. Suppose that $A$ is a closed subset of $Y = Z \times [0, \Omega)$ and $f : A \to K$ is a map. Suppose further that

(*) there exists $0 \leq \lambda < \Omega$ such that if $\lambda \leq \mu < \Omega$ and $(a, \mu), (a, \beta) \in A$, then $f(a, \mu) = f(a, \beta)$.

Then $f$ extends to a map of $Y$ to $K$.

Proof: Since $(Z \times \{0\})\pi K$ and $Z \times \{0\}$ is closed in $Y$, we may as well assume that $Z \times \{0\} \subset A$. Let $\pi_\lambda : Z \times [\lambda, \Omega) \to Z \times \{\lambda\}$ be the map given by $\pi_\lambda(z, \alpha) = (z, \lambda)$. Noting that $[\lambda, \Omega)$ is homeomorphic to $[0, \Omega)$, one may apply Lemma 3.6 to see that $\pi_\lambda$ is a closed map. Let $A^\# = A \cap (Z \times [\lambda, \Omega))$, $\pi^\# = \pi_\lambda|A^\# : A^\# \to Z \times \{\lambda\}$, and $A_\lambda = \pi^\#(A^\#)$. Then, of course, $A_\lambda$ is a closed subset of both $Z \times \{\lambda\}$ and $Z \times [0, \lambda]$.

Now $\pi^\# : A^\# \to A_\lambda$ is a closed map and hence is a quotient map. As a result of this and (*), there is a map $f_\lambda : A_\lambda \to K$ such that if $(z, \lambda) \in A_\lambda$ and $(z, \lambda) = \pi^\#(z, \alpha)$ with $(z, \alpha) \in A^\#$, then $f_\lambda(z, \lambda) = f(z, \alpha)$.

Put $h_0 = f|(A \cap (Z \times [0, \lambda]))$. Then $h_0|(A \cap A_\lambda) = f_\lambda|(A \cap A_\lambda)$. This shows that there is a map $h_1 : (A \cap (Z \times [0, \lambda])) \cup A_\lambda \to K$ such that $h_1|(A \cap (Z \times [0, \lambda])) = f|(A \cap (Z \times [0, \lambda]))$ and $h_1|A_\lambda = f_\lambda$.

Applying Lemma 3.4(2) and Theorem 3.1, one sees that $(Z \times [0, \lambda])\pi K$. Since $(A \cap (Z \times [0, \lambda])) \cup A_\lambda$ is closed in $Z \times [0, \lambda]$, then there is a map $F : Z \times [0, \lambda] \to K$ having the property that $F(t) = h_1(t)$ for all $t \in (A \cap (Z \times [0, \lambda])) \cup A_\lambda$. Let $q : [0, \Omega) \to [0, \lambda]$ be the unique retraction sending $\mu$ to $\lambda$ for all $\mu > \lambda$ and $r = \text{id}_Z \times q : Y \to Z \times [0, \lambda]$. Then for each $z \in Z$ and $\lambda \leq \mu < \Omega$, $r(z, \mu) = (z, \lambda)$. Thus, $F \circ r : Y \to K$ is a map, and in consideration of the preceding construction, one can check that $(F \circ r)|A = f$. □

Lemma 3.8. Let $Z$ be a compact metrizable space. Then $Y = Z \times [0, \Omega)$ is normal.

Proof: Let $A \subset Y$ be a closed subset and $f : A \to \mathbb{R}$ a map. For each $n \in \mathbb{N}$, apply Lemma 3.5 to find $\lambda_n \in [0, \Omega)$ so that

(*) if $\lambda_n \leq \mu < \beta < \Omega$ and $(z, \mu), (z, \beta) \in A$, then $|f(z, \mu) - f(z, \beta)| < \frac{1}{n}$.

Put $\lambda = \sup\{\lambda_n \mid n \in \mathbb{N}\}$. Then $\lambda \in [0, \Omega)$, and

(*) if $\lambda \leq \mu < \beta < \Omega$ and $(z, \mu), (z, \beta) \in A$, then $|f(z, \mu) - f(z, \beta)| = 0$. 34

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By Lemma 3.7 the map $f$ extends to a map of $Y$ to $\mathbb{R}$. □

**Corollary 3.9.** For every compact metrizable space $Z$, $Z \times [0, \Omega)$ is binormal. □

In [5, Proposition 4.4], the authors prove that if $X$ is a binormal pseudo-compact space, then $X$ has the homotopy extension property with respect to CW-complexes. Applying Lemma 2.1(1) and Corollary 3.9, we see the following corollary.

**Corollary 3.10.** Let $Z$ be a compact metrizable space. Then $Z \times [0, \Omega)$ has the homotopy extension property with respect to CW-complexes. □

Now we have our main theorem.

**Theorem 3.11.** Let $Z$ be a compact metrizable space and $Y = Z \times [0, \Omega)$.

(1) Then $Y$ is pseudo-compact, Hausdorff, and binormal, and

(2) if $K$ a CW-complex and $Z \tau K$, then both $Y \tau K$ and $\beta(Y) \tau K$.

**Proof:** (1) Certainly $Y$ is Hausdorff. Its pseudo-compactness follows from Lemma 3.4(4) and Lemma 2.1(1), while Corollary 3.9 yields its binormality.

(2) Let $A$ be closed in $Y$ and $f : A \to K$ be a map. Since $Y$ is normal and $A$ is closed, Lemma 2.1(2) yields that $A$ is pseudo-compact. Applying Lemma 2.2, we find a finite subcomplex $L$ of $K$ such that $f(A) \subset L$. Since $L$ is a finite dimensional metrizable compactum, we may assume that $L \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $1 \leq k \leq n$ and $\pi_k : \mathbb{R}^n \to \mathbb{R}$ be the $k$-coordinate projection. Applying Lemma 3.5, we find $\lambda_k \in [0, \Omega)$ such that if $\lambda_k \leq \mu < \beta < \Omega$, and $(a, \mu), (a, \beta) \in A$, then $\pi_k \circ f(a, \mu) = \pi_k \circ f(a, \beta)$. Let $\lambda = \max\{\lambda_k | 1 \leq k \leq n\}$. Then $\lambda \in [0, \Omega)$. We deduce from the preceding that $(\ast)$ of Lemma 3.7 is in effect for this choice of $\lambda$. Therefore, $f$ extends to a map of $Y$ to $K$. The final statement follows from this and Proposition 2.4. □

It is well known that $\beta([0, \Omega)) \cong [0, \Omega]$. Here is our final result.

**Corollary 3.12.** Let $K$ be a nonempty CW-complex. Then both $[0, \Omega) \tau K$ and $[0, \Omega] \tau K$.

**Proof:** Let $Z = \{0\}$, and apply Theorem 3.11(2). □
References


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