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by

LEANDRO F. AURICHI

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Department of Mathematics & Statistics

Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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***D*-SPACES, TOPOLOGICAL GAMES, AND SELECTION PRINCIPLES**

LEANDRO F. AURICHI

ABSTRACT. We present some connections between selection principles and the *D*-space problem. Our main result is that every Menger space is a *D*-space. We also present a study of the Rothberger property restricted to compact spaces.

1. INTRODUCTION

The definition of a *D*-space appeared in [7]. There, it was also asked if it is true that every Lindelöf space is a *D*-space. Many classes of spaces were proven to be included in the class of *D*-spaces (see, e.g., [1], [6], [5], [9]), but the question about Lindelöf spaces is still open. In [8], it was said that one of the main problems about this question is the lack of covering properties that imply being a *D*-space. In this work, we try to make a step in this direction, proving that a selection principle for open covers implies the *D* property. Such selection principles have been extensively studied, see, e.g., [16], [4], thus these results can give some new light to this

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problem. Following this idea, we present some topological games that are related to selection principles and give the relation between them and D -spaces.

In [3], it was shown that, under MA, Lindelöf spaces that are a union of fewer than continuum compact spaces are D -spaces. This can be obtained by our main result simply by using the fact that such spaces have the Menger property.

All the definitions about selection principles can be found in [16].

We finish this section defining D -space, fixing some notation, and proving a simple result.

Definition 1.1. Let X be a topological space. We say that $(V_x)_{x \in X}$ is an *open neighborhood assignment* for X if each V_x is an open neighborhood of x . If $Y \subset X$, we denote by V_Y the set $\bigcup_{y \in Y} V_y$. We say that Y is a *kernel* for $(V_x)_{x \in X}$ if $V_Y = X$. Finally, we say that X is a *D -space* if, for every open neighborhood assignment $(V_x)_{x \in X}$, there is a closed discrete kernel.

In [5], it was proven that every strong Σ -space is a D -space. In particular, this result implies that every Moore space is a D -space. A Moore space is a regular developable space. We will present here a direct proof which also shows that the regularity in this case is not needed. Before this, we give some definitions.

Definition 1.2. Let X be a topological space, \mathcal{C} a family of open sets, and $x \in X$. We use $st(x, \mathcal{C})$ to denote the set $\bigcup\{C \in \mathcal{C} : x \in C\}$. We say that X is a *developable space* if there is a sequence of open covers $(\mathcal{C}_n)_{n \in \omega}$ such that for each $x \in X$, $\{st(x, \mathcal{C}_n) : n \in \omega\}$ is a local base for it.

Proposition 1.3. *Every developable space is a D -space.*

Proof: Let X be a developable space and let $(\mathcal{C}_n)_{n \in \omega}$ be a sequence of covers such that for each $x \in X$, $\{st(x, \mathcal{C}_n) : n \in \omega\}$ is a local base for it. Let $(V_x)_{x \in X}$ be an open neighborhood assignment for X . For each $n \in \omega$, let $X_n = \{x \in X : st(x, \mathcal{C}_n) \subset V_x\}$. This set can be empty, but note that $X = \bigcup_{n \in \omega} X_n$. For each $n \in \omega$, we will define $D_n \subset X_n$ satisfying the following:

- (a) D_n is closed discrete;
- (b) $V_{D_n} \cup \bigcup_{k < n} V_{D_k} \supset X_n$.

Well order $X_n \setminus \bigcup_{k < n} V_{D_k}$ and define d_ξ as the least element of $(X_n \setminus \bigcup_{k < n} V_{D_k}) \setminus V_{E_\xi}$ where $E_\xi = \{d_\eta : \eta < \xi\}$. Define D_n as the set of such d_ξ 's. We will prove that D_n is closed discrete. Let $y \in X$. Let $C \in \mathcal{C}_n$ such that $y \in C$. Suppose there is a $d_\xi \in C$. Pick the least one. Note that $y \in st(d_\xi, C_n)$. Thus, $y \in V_{d_\xi} \cap C$ and $V_{d_\xi} \cap C \cap D_n = \{d_\xi\}$.

Therefore, $D = \bigcup_{n \in \omega} D_n$ is such that $V_D = X$ and it is closed discrete. \square

2. THE PARTIAL OPEN NEIGHBORHOOD ASSIGNMENT GAME

Through this work we will deal with some games that are played by two players. In general, one of the players is looking for a cover of a space and the other one is not. For mnemonic reasons, we will name these players C and N , respectively.

The general idea for the first game is the following: player N gives a partial open neighborhood assignment for the space that is enough to cover it. Then player C chooses a subset of the domain of this partial open neighborhood assignment that will be a part of the closed discrete kernel that C is looking for. Then player N chooses another partial open neighborhood assignment and player C chooses another part for its closed discrete kernel.

Definition 2.1. Let X be a topological space. We call the following game between players N and C the *partial open neighborhood assignment game (PONAG)*. Player N defines a partial open neighborhood assignment $(V_x)_{x \in Y_0}$ for X with $Y_0 \subset X$ and such that $V_{Y_0} = X$. After this, player C chooses $D_0 \subset Y_0$, a closed discrete subset of X . Then, at the n -th inning,

- N chooses $(V_x)_{x \in Y_n}$, a partial open neighborhood assignment compatible with every $(V_x)_{x \in Y_k}$ for $k < n$ such that
 - it covers $X \setminus \bigcup_{k < n} V_{D_k}$ and
 - $Y_n \cap \bigcup_{k < n} V_{D_k} = \emptyset$;
- C chooses $D_n \subset Y_n$, a closed discrete subset of X .

We say that C wins the game if $\bigcup_{n \in \omega} V_{D_n} = X$.

One of the advantages of this game is that we do not have to care about making the kernel closed discrete.

Proposition 2.2. *Let $(D_n)_{n \in \omega}$ be as in a PONAG won by player C . Then $\bigcup_{n \in \omega} D_n$ is a closed discrete set.*

Proof: Let $x \in X$, and let $n \in \omega$ be the first such that $x \in \bigcup_{d \in D_n} V_d$. Note that $\bigcup_{d \in D_n} V_d$ is an open set that separates x from every point in $\bigcup_{k > n} D_k$. Since $\bigcup_{k \leq n} D_k$ is closed discrete (it is a finite union of closed discrete sets), x is not an accumulation point of $\bigcup_{k \in \omega} D_k$. \square

It is easy to see that being a D -space is related to player N not having a winning strategy.

Proposition 2.3. *Let X be a topological space. If N has no winning strategy for the PONAG, then X is a D -space.*

Proof: Suppose X is not a D -space. Let $(V_x)_{x \in X}$ be an open neighborhood assignment for X that witnesses it. Then, if N plays at every inning n , $(V_x)_{x \in A}$ where $A = X \setminus \bigcup_{k < n} V_{D_k}$, C cannot win this game. \square

It is easy to prove that for a σ -compact space, player C has a winning strategy for the PONAG. There is a generalization for σ -compactness that we can prove which implies that player N has no winning strategy for the PONAG. Before proving this, we give some definitions.

Definition 2.4. We say that a space X is a *Menger space* if for every sequence $(\mathcal{U}_n)_{n \in \omega}$ of open covers there is $(U_n)_{n \in \omega}$ such that each U_n is a finite subset of \mathcal{U}_n and $\bigcup_{n \in \omega} U_n = X$.

Note that the Menger property implies the Lindelöf property; thus, we can assume that each cover in the definition of the Menger property is countable. Also, it is easy to see that the previous definition has an equivalent formulation if we also require each \mathcal{U}_n to be closed under finite unions and each U_n to be a unitary subset of \mathcal{U}_n .

The property of being a Menger space has an equivalent formulation using a game.

Definition 2.5. Let X be a Lindelöf space. We call the *Menger game* the following game played between players N and C . At each inning $n \in \omega$, player N chooses a countable open cover \mathcal{U}_n that is closed under finite unions. Then player C chooses $U_n \in \mathcal{U}_n$. We say that player C wins the game if $\bigcup_{n \in \omega} U_n = X$.

In [11], it was proven that a Lindelöf space is a Menger space if and only if player N has no winning strategy for the Menger game. With this result, we can prove the following proposition.

Proposition 2.6. *Let X be a Menger space. Then N has no winning strategy for the PONAG.*

Proof: Suppose that N has a winning strategy for the PONAG. We will define a winning strategy for N for the Menger game. Since X is Menger, this will give us a contradiction. At the first inning, N chooses $\mathcal{U}_0 = \{V_x : x \in Y_0\}$ where $(V_x)_{x \in Y_0}$ is the one chosen by N at the first play in the winning strategy for the PONAG. If $(V_x)_{x \in U_0}$ is the finite subset of $(V_x)_{x \in Y_0}$ played by C in the Menger game, then consider C in PONAG as playing the finite set $U_0 \subset Y_0$. Then at the second inning, N chooses $\mathcal{U}_1 = \{V_{U_0} \cup V_x : x \in Y_1\}$ where Y_1 is the one that N chooses in the winning strategy for the PONAG after C chooses $(V_x)_{x \in U_0}$. Note that proceeding this way, N wins the game. \square

Corollary 2.7. *Every Menger space is a D -space.*

But the PONAG does not give us an equivalence for being a D -space. The space of the irrationals with the usual topology is a D -space (see, e.g., [1]), and we can prove the following proposition.

Proposition 2.8. *In the PONAG for ω^ω , player N has a winning strategy.*

Proof: For each $n \in \omega$, let $A_n = \{y_j^n \in \omega^\omega : j \in \omega\}$ and $(V_{y_j^n})_{j \in \omega}$ be such that

- $y_j^n(m) = \begin{cases} n + 1 & \text{if } m \leq j \\ 0 & \text{otherwise} \end{cases}$
- $V_{y_j^n} = \{x \in \omega^\omega : x(n) \leq n + 1 + j\}$.

Note that $A_n \cap A_k = \emptyset$ if $n \neq k$ and $y_j^n \in V_{y_j^k}$ for any $j, n \in \omega$. Note also that $(y_j^n)_{j \in \omega}$ is a converging sequence to the function constantly equal to $n + 1$.

At the first inning, N chooses $V_{y \in Y_0}$ where $Y_0 = A_0$. Note that player C can choose only finitely many points of Y_0 . Let U_0 be the y_j^0 's chosen. Let $k = \max\{j + 1 : y_j^0 \in U_0\}$. Then player N chooses $Y_1 = A_{k+1}$. Note that $V_{Y_1} = \omega^\omega$ and $Y_1 \cap V_{U_0} = \emptyset$. Again, player C can choose only finitely many points in Y_1 . Let U_1 be the set

of the y_j^{k+1} 's chosen. Let $k' = \max\{j + k + 1 : y_j^{k+1} \in U_1\}$. Then player N chooses $Y_2 = A_{k'+1}$. Proceeding this way, player N wins the game. \square

3. THE STAR GAME

The next game is based on the proof in [1] that every space with a point countable base is a D -space.

Definition 3.1. Let X be a topological space. We call the *star game* the following game played between C and N . At each inning ξ , player C chooses x_ξ different from every x_α chosen before. Then player N chooses $S_\xi \subset X$ such that $x_\xi \in S_\xi$ and for every $y \in S_\xi$, N chooses an open set V_y such that $y, x_\xi \in V_y$. Also, if $y \in S_\eta$ for some $\eta < \xi$, then V_y has to be the same one previously chosen. At the beginning of every inning, the following tests are made in this order:

- (a) If there is an $x \in X$ such that x is an accumulation point for $\{x_\eta : \eta < \xi\}$ and x is not an element of any S_η with $\eta < \xi$, then N loses. We call this condition S .
- (b) If $\{x_\eta : \eta < \xi\}$ is not a closed discrete set, then C loses.
- (c) If $\bigcup_{\eta < \xi} V_{x_\eta} = X$, then C wins. Otherwise, the game continues.

Proposition 3.2. *Let X be a topological space. If N has no winning strategy for the star game, then X is a D -space.*

Proof: Suppose that X is not a D -space. Let $(V_x)_{x \in X}$ be an open neighborhood assignment witnessing it. If N plays at every inning ξ , $S_\xi = \{y : x_\xi \in V_y\}$, then player N wins. \square

For σ -compact spaces, the situation is similar to the PONAG.

Proposition 3.3. *If X is a σ -compact space, then player C has a winning strategy for the star game.*

Proof: First, suppose X is compact. Player C plays as follows. Define x_0 as any point of X . Next, define x_{n+1} as any point (if there is any) not in $\bigcup_{m < n} V_{S_m}$. Note that if this procedure continues until the ω -th inning, then any accumulation point of $\{x_n : n \in \omega\}$ must be outside $\bigcup_{n \in \omega} S_n$. Thus, unless player N loses because of condition S , there is an $n \in \omega$ such that $\bigcup_{m < n} V_{S_m}$ covers X . Since

X is compact, we can find a finite subset F of $\bigcup_{m < n} S_m \setminus \{x_m : m < n\}$ such that $V_F \cup V_{\{x_m : m < n\}}$ covers X .

For proving the σ -compact case, just write $X = \bigcup_{n \in \omega} K_n$ with each K_n compact and repeat the strategy above countably many times. \square

It is not difficult to see that a stronger version of the Menger property implies that player N has no winning strategy for the star game. Before proving this, let us give some definitions.

Definition 3.4. We say that a space X is a *Rothberger space* if for every sequence $(\mathcal{U}_n)_{n \in \omega}$ of open covers of X there is a cover $(U_n)_{n \in \omega}$ for X such that each $U_n \in \mathcal{U}_n$.

Note that the Rothberger property implies Lindelöfness. Thus, we can assume again that all the covers are countable. Note also that the Rothberger property implies the Menger property. Similar to the Menger property, the Rothberger property also has a characterization using a game.

Definition 3.5. Let X be a topological space. The *point-open game* is the following game played between players N and C . At each inning $n \in \omega$, player C chooses $x_n \in X$ and player N chooses an open neighborhood V_n for x_n . Player C wins if $\bigcup_{n \in \omega} V_n = X$.

In [14], it was proven that being a Rothberger space is equivalent to player N not having a winning strategy for the point-open game. Therefore, if X is a Rothberger space, player N cannot have a winning strategy for the star game, because, until the ω -th inning, this game is played like the point-open game, just with more restrictions for player N .

Proposition 3.6. *If X is a Rothberger space, then player N has no winning strategy for the star game.*

In [1], in the proof that every space with a point countable base is a D -space, actually it was proven that a more general class of spaces (a simple generalization of the concept of having an uniform base) is D .

Definition 3.7. We say that a base \mathcal{B} for a topological space X is ω -uniform if for every $x \in X$ and every $B \in \mathcal{B}$ such that $x \in B$, the set $\{A \in \mathcal{B} : x \in A \text{ and } A \not\subset B\}$ is countable.

Note that any point countable base is an ω -uniform base. But the converse is not true, even for Lindelöf spaces: let D be a discrete space of cardinality ω_1 and add a point ∞ such that a local base for it is made up of sets of the form $V \cup \{\infty\}$ with $D \setminus V$ countable. The space $D \cup \{\infty\}$ is a Lindelöf space with an ω -uniform base but without a point countable base.

We will prove that having an ω -uniform base is enough for a space to have a winning strategy for player C for the star game. Before this, we need some more definitions.

Definition 3.8. Let X be a topological space. We say that an inning α for the star game is a *closed inning* if

- $D_\alpha = \{x_\xi : \xi < \alpha\}$ is a closed discrete set in X ;
- $x_\xi \notin V_{x_\eta}$ for any $\eta < \xi < \alpha$;
- $V_{D_\alpha} \supset \bigcup_{\xi < \alpha} S_\xi$ (S_ξ is the one in Definition 3.1).

We say that player C has a *partial strategy* for the star game if, for every closed inning α and every $x \notin \bigcup_{\xi < \alpha} V_{x_\xi}$, there is a closed inning β such that $\{x_\xi : \alpha \leq \xi < \beta\} \subset X \setminus \bigcup_{\xi < \alpha} V_{x_\xi}$ and $x_\alpha = x$.

The idea of the closed inning is to split a strategy into smaller parts.

Proposition 3.9. *Let X be a topological space. If player C has a partial strategy for the star game, then player C has a winning strategy for the star game.*

Proof: The strategy for player C will simply be a sequence of partial strategies concatenated. At each inning α , player C does the following:

- if C is in the middle of a partial strategy, C continues it;
- otherwise, we assume that α is a closed inning. If $\bigcup_{\xi < \alpha} V_{x_\xi} = X$, the game is finished. Otherwise, player C begins a new partial strategy for an $x \notin \bigcup_{\xi < \alpha} V_{x_\xi}$.

Let us prove that this is a winning strategy. For this, it is enough to show that a limit of closed innings is also a closed inning. Let α be a limit of closed innings and let $x \in \overline{\{x_\xi : \xi < \alpha\}}$. We can suppose that $x \in S_\xi$ for some $\xi < \alpha$ (otherwise, player N loses). Thus, there is a $\beta < \alpha$ such that $x \in V_{x_\beta}$ (because of the partial strategies). We may assume by induction that $\{x_\xi : \xi < \beta\}$ is closed discrete.

Since $x_\eta \notin V_{x_\beta}$ for $\eta > \beta$, x cannot be an accumulation point of $\{x_\xi : \xi < \alpha\}$. \square

Now we just have to prove that having an ω -uniform base is enough to have a partial strategy.

Proposition 3.10. *If X has an ω -uniform base, then player C has a partial strategy for the star game.*

Proof: Let \mathcal{B} be an ω -uniform base for X . Since having an ω -uniform base is hereditary, we just have to prove that for any x there is a closed inning that covers x . We may suppose that all V_x belong to \mathcal{B} . Let $x \in X$. We will define a strategy for player C such that ω will be a closed inning. For technical reasons, we will enumerate the innings for this strategy using numbers of the form p^n where p is a prime number and $n > 0$. Define $x_2 = x$. After player N plays, enumerate the set $\{U \in \mathcal{B} : \exists y \in S_2 \ x_2 \in V_y \text{ and } U \setminus V_{x_2} \neq \emptyset\}$ as $(U_n^{x_2})_{n \in \omega}$. Assume that for each $k < n$, with $k = p^m$ for some prime p , player C already chose x_k and enumerated the set $\{U \in \mathcal{B} : \exists y \in S_0 \ x_k \in V_y \text{ and } U \setminus V_{x_k} \neq \emptyset\}$ as $(U_n^{x_k})_{n \in \omega}$. At the n -th inning, with $n = p^m$ for some prime p , player C does the following. Let q be the m -th number of the form r^s with r being a prime number. If there is an $x \in \bigcup_{k < n} S_k$ such that

$$(3.1) \quad V_x \in \{U_t^{x_q} : t \in \omega\} \text{ and } x \notin \bigcup_{k < n} V_{x_k},$$

then player C chooses x_n as the x such that $V_x = U_t^{x_q}$ with t being the least one satisfying (3.1). Enumerate $(U_t^{x_n})_{t \in \omega}$ as usual. If there is no such x satisfying (3.1), proceed to the next inning without choosing an x (actually, the definition of the game does not allow this, but there is no problem if we just assume that player N will do nothing if player C does nothing).

Let us prove that this is a partial strategy. We begin proving that $\bigcup_{n \in \omega} S_n \subset \bigcup_{n < \omega} V_{x_n}$. Let $x \in S_n$ for some $n \in \omega$. Assume that n is the least one with this property. Then $V_x = U_k^{x_n}$ for some $k \in \omega$. Note that in our construction, at every inning of the form p^t , where p is a prime number and t indicates how many elements there are before n in our indexes, we pick the first $U_s^{x_n}$ that has some element not covered. Since there are only finitely many elements before k , $U_k^{x_n}$ will be covered in some inning and,

therefore, x . Now we have to prove that $\{x_n : n \in \omega\}$ is closed discrete in $\bigcup_{n \in \omega} S_n$. Let $x \in \bigcup_{n \in \omega} S_n$. Let n be the first one such that $x \in V_{x_n}$. Since there are only finitely many x_k 's with $k < n$ and no $x_m \in V_{x_n}$ for $m > n$, we are done. \square

Corollary 3.11. *If X has an ω -uniform base, then player C has a winning strategy for the star game. Therefore, X is a D -space.*

The relation between the star game and the D property is not clear. We do not have any example of a D -space such that player N has a winning strategy for the star game. Since it is not known if a hereditarily Lindelöf space must be a D -space, the following question is particularly interesting.

Question 3.12. If X is a hereditarily Lindelöf space, can player N have a winning strategy for the star game?

4. SOME APPLICATIONS

It is easy to see that if X is a Menger space and \mathbb{P} is a countably closed forcing, then if \mathbb{P} preserves the Lindelöfness of X , it has also to preserve its property of being Menger. We can prove an analogous result for D -spaces such that N has no winning strategy for the PONAG. Before proving this, we prove a lemma.

Lemma 4.1. *Let X be a topological space and \mathbb{P} be a countably closed forcing. Let $p \in \mathbb{P}$ and $(\check{V}_x)_{x \in X}$ be such that $p \Vdash \text{“}(V_x)_{x \in X}$ is an open neighborhood assignment for \check{X} ” and $p \Vdash \check{X}$ is Lindelöf. Let $U \subset X$ be an open set. Then there are $q \leq p$, a countable $A \subset X \setminus U$, and $(V_x)_{x \in A}$ such that $q \Vdash \check{V}_A \cup \check{U} = \check{X}$ and $\check{V}_x = \dot{V}_x$ for every $x \in A$.”*

Proof: Note that $p \Vdash \check{X} \setminus U$ is Lindelöf. Thus, $p \Vdash \text{“there is a countable } \dot{A} \text{ such that } V_{\dot{A}} \supset \check{X} \setminus \check{U} \text{ and } \dot{A} \cap \check{U} = \emptyset \text{.”}$ Let $q' \leq p$ be such that q' decides \dot{A} and let $q \leq q'$ be such that q decides $(\dot{V}_x)_{x \in A}$. \square

Proposition 4.2. *Let X be a topological space such that N has no winning strategy for the PONAG. Let \mathbb{P} be a countably closed forcing such that $\mathbb{P} \Vdash \check{X}$ is Lindelöf. Then $\mathbb{P} \Vdash \check{X}$ is a D -space.*

Proof: We will define a strategy for N and, for each possible move of N , we will associate a $p \in \mathbb{P}$ in the following way:

- In the first inning, N chooses $(V_x)_{x \in A}$, where A is the one given by the lemma applied to 1 and the empty set. The q given by the lemma is the element associated with this play.
- Let $(V_x)_{x \in Y}$ be the move of N at the n -th inning. Let p be the element associated with this play and $D \subset Y$ be the closed discrete set chosen by player C . At the $(n + 1)$ -th inning, player N chooses $(V_x)_{x \in A}$ where A is the one given by the lemma applied to p and U , where U is the union of every V_D already played by player C . The forcing condition associated with this move is the $q \leq p$ given by the lemma.

Since N has no winning strategy, there is $(D_n)_{n \in \omega}$ where D_n is the one chosen by C at the n -th inning such that $\bigcup_{n \in \omega} V_{D_n} = X$. Note that $D = \bigcup_{n \in \omega} D_n$ is closed discrete. Let $(p_n)_{n \in \omega}$ be the forcing conditions associated with every move of N in this game. Note that $p_{n+1} \leq p_n$. Let $p \leq p_n$ for every n . Note that $p \Vdash \check{V}_D = X$ and $\check{V}_x = \check{V}_x$ for every $x \in \check{D}$. So p forces that X is a D -space. \square

One of the difficulties in proving that a space X is a D -space is determining where to look for the closed discrete kernel. For a fixed open neighborhood assignment $(V_x)_{x \in X}$, one can easily fix a countable set to look at if X is Lindelöf. Namely, pick M a countable elementary submodel such that $(V_x)_{x \in X} \in M$. It is easy to see that if $(V_x)_{x \in X}$ has a closed discrete kernel, then it has one as a subset of $M \cap X$. But if we assume that X is a Menger space, we can do a little better.

Proposition 4.3. *Let X be a Menger space and $(V_x)_{x \in X}$ be an open neighborhood assignment for it. If $Y \subset X$ is such that for every finite $F \subset Y$, $V_F \cup V_{Y \setminus V_F} = X$, then there is a closed discrete (in X) $D \subset Y$ such that $V_D = X$.*

Proof: Suppose not. Consider the following strategy for player N for the Menger game. At each inning, player N defines $\mathcal{U}_n = \{\bigcup_{m < n} U_m \cup V_F : F \subset Y \setminus \bigcup_{m < n} U_m \text{ is finite}\}$ where the U_m 's are the plays of player C . Since each U_m is a finite union of V_x 's with $x \in Y$ and by our hypothesis over Y , \mathcal{U}_n is a cover for X . As in the case of the PONAG, note that if $\bigcup_{n \in \omega} U_n = X$, then $\bigcup_{n \in \omega} F_n$, where each $U_n = V_{F_n}$, is a closed discrete set. Since there is no

closed discrete set inside Y that covers X , then player N wins the game, contradicting the fact that X is a Menger space. \square

Showing that a given space is a Menger space is often easier than proving it is a D -space directly. One example is a Souslin tree with the fine wedge topology or the coarse wedge topology [2]. Such spaces are easily shown to be Rothberger by a combinatorial argument. Therefore, they are Menger spaces and D -spaces. This is an easier proof than the one presented in [15], where the D property is shown directly.

Also, proving that a space is a D -space by proving it is a Menger space is a technique that applies to a wider class of spaces than the class of metrizable spaces or σ -compact spaces, as we can see in the next example.

Example 4.4. Let X be a Menger space and κ be any uncountable cardinal. Define Y' as the disjoint union of κ many copies of X and let Y be a one-point Lindelöfication of Y' . Note that Y is a Menger space and it is neither metrizable nor σ -compact.

5. COMPACT ROTHBERGER SPACES

The Rothberger property restricted to compact spaces has several characterizations. Some of them are well known and others were presented somewhere else. For the convenience of the reader, we will present here a list of some of them, putting together new and old ones. Before this, we need the following definitions.

Definition 5.1. Let X be a Lindelöf space. We call the *open-open game* the following game played between players N and C . At each inning $n \in \omega$, player N chooses \mathcal{U}_n , a countable open cover for X . Then player C chooses $U_n \in \mathcal{U}_n$. We say that player C wins the game if $\bigcup_{n \in \omega} U_n = X$.

In [14], it was proven that X is Rothberger if and only if player N has no winning strategy for the open-open game.

In [17], a characterization for preservation of the Lindelöfness of a space under countably closed forcing was proven. This characterization uses the concept of a covering tree and, using the characterization of being a Rothberger space by topological games, it is simple to see that Rothberger spaces have the preservation property. It

turns out that the same technique gives us a characterization in the class of compact spaces and preservation under any forcing. This result also can be proven using a combination of older results, but this proof seems to have its own interest. The next definitions are versions of concepts presented in [17].

Definition 5.2. Let X be a compact space. We call a *covering ω -tree* a family $(V_s)_{s \in \omega^{<\omega}}$ where each V_s is an open set of X and for every $s \in \omega^{<\omega}$, $\{V_{s \smallfrown n} : n \in \omega\}$ is a cover for X . The order on $\{V_s : s \in \omega^{<\omega}\}$ is the one induced by the usual one on $\omega^{<\omega}$.

Definition 5.3. Let X be a compact space and $(V_s)_{s \in \omega^{<\omega}}$ be a covering ω -tree for X . We say that V_t with $t \in \omega^{<\omega}$ is a *covering element* for $(V_s)_{s \in \omega^{<\omega}}$ if $\bigcup_{a \leq t} V_a \supset X$.

There is a characterization using elementary submodels and a special space defined using them [13].

Definition 5.4. Let (X, τ) be a topological space and M be an elementary submodel such that $(X, \tau) \in M$. We call X_M the space $X \cap M$ with the topology generated by $\{U \cap M : U \in \tau \cap M\}$.

Proposition 5.5. *Let X be a compact Hausdorff space. The following are equivalent.*

- (a) X is scattered;
- (b) X is Rothberger;
- (c) the compactness of X is preserved by any forcing;
- (d) the compactness of X is preserved by adjoining a Cohen real;
- (e) for every covering ω -tree for X , the set of the covering elements is dense;
- (f) for every covering ω -tree for X , there is a covering element;
- (g) there is a countable elementary submodel M such that X_M is compact Hausdorff;
- (h) X_M is compact Hausdorff for any elementary submodel M such that $X \in M$.

Proof: (c) \Rightarrow (d) is trivial.

For (d) \Rightarrow (c), let \mathbb{P} be a forcing and let \dot{C} be a name for an open cover for X such that $\mathbb{P} \Vdash \text{“}\dot{C} \text{ has no finite subcover.”}$ We may assume as well that \mathbb{P} forces that every element of \dot{C} is an

open set of the ground model. We construct a tree $((p_s, V_s))_{s \in \omega^{<\omega}}$ such that, for every $s, t \in \omega^{<\omega}$,

- (i) $p_s \in \mathbb{P}$ and $p_s \leq p_t$ if $s \supset t$;
- (ii) $p_s \Vdash V_s \in \dot{C}$;
- (iii) $\{V_{s \frown n} : n \in \omega\}$ is a cover for X .

Note that $((p_s, V_s))_{s \in \omega^{<\omega}}$ is isomorphic to Cohen forcing. Let g be a Cohen real in the extension after we force with such order. Note that by genericity $\bigcup_{s \subset g} V_s$ is an open cover for X . Since we have (d), there is a finite subcover for it, although \mathbb{P} forces that there is not.

For (d) \Rightarrow (e), let $(V_s)_{s \in \omega^{<\omega}}$ be a covering ω -tree for X . Suppose that there is an $s \in \omega^{<\omega}$ such that there is no $t \supset s$ such that V_t is a covering element. Let g be a Cohen real extending s . By (d) there is a $s \subset t \subset g$ such that V_s is a covering element.

For (e) \Rightarrow (d), suppose that Cohen forcing destroys X . Proceed as we did in the proof of (d) \Rightarrow (c) and construct a covering ω -tree. Since we have (e), any generic cover has a finite subcover.

(e) \Rightarrow (f) is immediate.

For (f) \Rightarrow (e), let $(V_s)_{s \in \omega^{<\omega}}$ be a covering ω -tree and let $t \in \omega^{<\omega}$. Note that $\{V_s : s \in \omega^{<\omega}, s \supseteq t\}$ is isomorphic to a covering ω -tree for X and, by (f), it has a covering element. Note that this covering element is an extension of V_t .

Note that a strategy for player N on the open-open game is a covering ω -tree and the converse is also true: any covering ω -tree is a strategy for the game. Also, note that such tree has a covering element if and only if the strategy associated is not a winning strategy. Thus, we have (b) \Leftrightarrow (f).

The equivalence among (a), (g), and (h) is done in [13].

It is well known that in the class of compact spaces the notions of being Rothberger and being scattered are equivalent. For the convenience of the reader, we present one proof of this result here.

(a) \Rightarrow (b): Suppose X is scattered. Let α be the height of X (see [10, p. 350] for the definitions of height and X^α). We will prove this by induction over α . Note that, since X is compact, $\alpha = \beta + 1$ with X^β finite. Let $(\mathcal{U}_n)_{n \in \omega}$ be open covers for X . Let $\{x_0, \dots, x_n\} = X^\beta$. Choose $U_k \in \mathcal{U}_k$ for $k = 0, \dots, n$ such that $x_k \in U_k$. Note that $X \setminus \bigcup_{k \leq n} U_k$ is a compact scattered space with

height less than β . Thus, by the induction hypothesis, there are $U_k \in \mathcal{U}_k$ with $k > n$ such that $\bigcup_{k>n} U_k \supset X \setminus \bigcup_{k \leq n} U_k$. Thus, X is Rothberger.

(b) \Rightarrow (a): We only have to prove that any closed subspace of X has an isolated point. Suppose not. Since any compact space without isolated points has 2^ω as a closed subspace, and 2^ω is easily seen not to be Rothberger, we have a contradiction. \square

There is a direct proof of (a) \Leftrightarrow (c) in [12].

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DEPARTAMENTO DE MATEMÁTICA; UNIVERSIDADE DE SÃO PAULO; CAIXA
POSTAL: 66281; SÃO PAULO, SP CEP: 05315-970, BRASIL
E-mail address: leandro.aurichi@gmail.com