CARDINALITY BOUNDS OF H-SETS IN URIYSOHN SPACES

by

DANIEL K. McNEILL

Electronically published on February 12, 2010
CARDINALITY BOUNDS OF H-SETS IN URYSOHN SPACES

DANIEL K. MCNEILL

Abstract. A Urysohn space $X$ is constructed which has an H-set $A$ with $|A| > 2^{|\psi(X)|}$, where $\psi(X)$ is the closed-pseudocharacter of the space $X$. The space provides a counterexample to Alessandro Fedeli’s question in \(\omega\)H-sets and cardinal invariants [Comment. Math. Univ. Carolin. 39 (1998), no. 2, 367–370]. In addition, it is demonstrated that there is no $\theta$-continuous map from a compact Hausdorff space to the space $X$ with the H-set $A$ as the image, giving a Urysohn counterexample to Johannes Vermeer’s conjecture in Closed subspaces of $H$-closed spaces [Pacific J. Math. 118 (1985), no. 1, 229–247]. Finally, it is shown that the cardinality of an H-set in a Urysohn space $X$ is bounded by $2^{\chi(X_s)}$, where $\chi(X)$ is the character of $X$ and $X_s$ is the semiregularization of $X$. This refines Angelo Bella’s result in A couple of questions concerning cardinal invariants [Questions Answers Gen. Topology 14 (1996), no. 2, 139–143] that the cardinality of such an H-set is bounded by $2^{\chi(X)}$.

Introduction

Recall that a space $X$ is H-closed if for every family of open sets covering $X$ there is a finite subfamily whose union is dense in $X$. More generally, an H-set of a space $X$ is a subset $A$ for which every family of open sets of $X$ covering $A$ has a finite subfamily whose
union is dense in $A$. Also recall the definitions of the pseudocharacter, closed-pseudocharacter, and character of a space. We will use the notation $\psi(X)$, $\overline{\psi}(X)$, and $\chi(X)$, respectively, for these three quantities. Finally also note that the semiregularization of a space $X$, denoted $X_\delta$, is the set $X$ with the topology generated by the regular open sets of $X$, $RO(X) = \{\text{int cl } U : U \text{ is open in } X\}$.

It is well known for a given space $X$ that $\psi(X) \leq \overline{\psi}(X) \leq \chi(X)$ (see [7]), in fact $\psi(X) \leq \overline{\psi}(X_\delta) = \overline{\psi}(X) \leq \chi(X)$. In particular, it follows that $2^{\psi(X)} \leq 2^{\overline{\psi}(X_\delta)} = 2^{\overline{\psi}(X)} \leq 2\chi(X) \leq 2\chi(X)$. If a statement $P$ is true for all $H$-closed spaces, a natural question to ask is whether $P$ also holds for $H$-sets, or perhaps for $H$-sets embedded in a space with a particular separation property. For example, Alan Dow and Jack R. Porter show in [5] that if $X$ is an $H$-closed space, then $|X| \leq 2^{\chi(X)}$ (in fact, they show $|X| \leq 2^{\psi(X)}$), and later, in [2], Angelo Bella shows that if $X$ is Urysohn and $A$ is an $H$-set of $X$, then $|A| \leq 2^{\chi(X)}$. On the other hand, Bella and I. V. Yaschenko show in [4] that an $H$-set $A$ of a Hausdorff space $X$ may have cardinality larger than $2^{\chi(X)}$. Similar results are those of Bella and Porter in [3] showing if $X$ is $H$-closed, then $|X|$ may be larger than $2^{\psi(X)}$; in fact, one example of this has $X = \kappa\omega$, a Urysohn space. Here a space $X$ is constructed demonstrating that the cardinality of an $H$-set of a Urysohn space is not always bounded by $2^{\overline{\psi}(X)}$. It is also proven that the cardinality of an $H$-set is bounded by $2^{\chi(X_\delta)}$. These two results constrain the maximum cardinality of an $H$-set of a Urysohn space as much as is possible with the inequalities listed above.

In [11], J. Vermeer conjectures that a subset $A$ of $X$ is an $H$-set if and only if there is a compact Hausdorff space $K$ and a $\theta$-continuous map $f : K \to X$ with $f[K] = A$. Bella and Yaschenko [4] provide a counterexample with countable character with their construction. A counterexample to Vermeer’s conjecture, similar in construction to Bella and Yaschenko’s, but which is Urysohn and has countable closed-pseudocharacter, is provided in this paper.

Bella and Yaschenko in [4] reintroduce, under the name of relatively $H$-closed, a concept first considered by P. Th. Lambrinos in [8] under the name of $H$-bounded and investigated further by Douglas D. Mooney in [9]. Given a space $X$, a subspace $A$ is called relatively $H$-closed if for every open cover $\mathcal{U}$ of $X$ there is a finite
subfamily of $\mathcal{U}$ whose union is dense in $A$. The following notation is introduced for the purpose of this paper.

**Notation 1.** Let $X \subseteq Y \subseteq Z$ be spaces. We write $H(X; Y; Z)$ if every cover $\mathcal{U}$ of $Y$ with open sets of $Z$ has a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ for which $X \subseteq \bigcup_{U \in \mathcal{F}} \text{cl}_Z U$.

When considering the space $\kappa X$ for a given space $X$, the author will use the notation $o(U) = U \cup \{p \in \kappa X \setminus X : U \in p\}$. The following lemma will also prove useful (see [10]).

**Lemma 2.** Let $X$ be a space. If $\mathcal{B}$ is an open neighborhood base of $x \in X$, then $\bigcap \{o(B) : B \in \mathcal{B}\} = \{x\}$.

Finally given a space $X$, we say an open filter $\mathcal{G} \subseteq \tau(X)$ has the weak countable intersection property if for every countable subset $\mathcal{G}'$ of $\mathcal{G}$ we have $\bigcap \{\text{cl}_X U : U \in \mathcal{G}'\} \neq \emptyset$. We then call a space $X$ weakly realcompact if every open ultrafilter $\mathcal{U} \subseteq \tau(X)$ with the weak countable intersection property has nonempty adherence.

1. **Constructions**

The first construction is a basic space from which the counterexample to Alessandro Fedeli’s question (see [6]) will be built. We modify a construction given by Bella and Yaschenko in [4].

**Construction 3.** Let $X$ be a weakly realcompact space with countable closed-pseudocharacter and $\kappa X$ Urysohn, e.g., $X = \omega$, and let $\tilde{X} = \kappa X \setminus X$. Define

$$Z = X \cup (X \times \omega \times \tilde{X}) \cup \tilde{X}$$

with the following topology. If $U \in \tau(X)$ and $n \in \omega$, then

$$U(n) = U \cup (U \times [n, \omega)) \times \tilde{X}) \in \tau(Z)$$

is a basic open neighborhood of $x \in X$, and if $p \in \tilde{X} = \kappa X \setminus X$ and $U \in p$, then a basic open neighborhood of $p$ is

$$U(p) = (U \times \omega \times \{p\}) \cup \{p\} \in \tau(Z).$$

Finally, the points of $X \times \omega \times \tilde{X}$ are isolated.

**Fact 4.** (1) $Z$ is Urysohn.

(2) $X$ is relatively $H$-closed in $Z$, in other words $H(X; Z; Z)$.

(3) $H(X; X \cup \tilde{X}; Z)$. 
(4) $Z$ has countable closed-pseudocharacter.

Proof: (1) If $x, y \in X$, then there exist $U$ and $V$ open neighborhoods in $X$, of $x$ and $y$, respectively, with $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Hence, $\text{cl}_Z U(0) \cap \text{cl}_Z V(0) = \emptyset$, and $U(0)$, and $V(0)$ are neighborhoods of $x$ and $y$ in $Z$.

If $p, q \in \hat{X}$, then there exist $U \in p$ and $V \in q$ such that $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Thus, $p \in U(p)$, $q \in V(q)$, and $\text{cl}_Z U(p) \cap \text{cl}_Z V(q) = \emptyset$.

If $x \in X$ and $p \in \hat{X}$, then there exist $U$ an open neighborhood of $x$ in $X$ and $V \in p$ with $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. Therefore, $x \in U(0)$, $p \in V(p)$, and $\text{cl}_Z U(0) \cap \text{cl}_Z V(p) = \emptyset$.

Finally, the points of $X \times \omega \times \hat{X}$ are isolated.

(2) Let $\mathcal{C}$ be an open cover of $Z$. Without loss of generality, for each $x \in X$ we can assume there exists an $U_x \in \tau(X)$ and an $n_x \in \omega$ with $x \in U_x(n_x) \in \mathcal{C}$; also we can assume that for each $p \in \hat{X}$ there exists a $V_p \in p$ with $V_p(p) \in \mathcal{C}$. Now, $\{U_x : x \in X\} \cup \{V_p \cup \{p\} : p \in \hat{X}\}$ is an open cover of $\kappa X$. Since $\kappa X$ is $H$-closed, there exist finitely many $x_1, x_2, \ldots, x_n$ and $p_1, \ldots, p_m$ such that $\kappa X = \bigcup_{i=1}^n \text{cl}_X U_{x_i} \cup \bigcup_{i=1}^m \text{cl}_X V_{p_i}$. Hence, $X \subseteq \bigcup_{i=1}^n \text{cl}_X U_{x_i} \cup \bigcup_{i=1}^m \text{cl}_X V_{p_i} \subseteq \bigcup_{i=1}^n \text{cl}_Z U_{x_i}(n_x_i) \cup \bigcup_{i=1}^m \text{cl}_Z V_{p_i}(p_i)$.

(3) Take a cover $\mathcal{U}$ of $X \times \hat{X}$ with open sets of $Z$. Now extend $\mathcal{U}$ to an open cover, $\mathcal{U}'$ of all of $Z$, by adding in the isolated singletons not already covered. Then there is a finite subfamily $\mathcal{V} \subset \mathcal{U}'$ with $X \subseteq \text{cl}_Z \cup \mathcal{V}$. However, it is clear that $\mathcal{V}$ need not contain any of the isolated singletons added to extend $\mathcal{U}$. Hence, we may take $\mathcal{V} \subset \mathcal{U}$.

(4) We must show every point in $Z$ is the intersection of a countable collection of closed neighborhoods. This is certainly true for the isolated points of $X \times \omega \times \hat{X}$. If, on the other hand, $x \in X \subset Z$, take $\{U_n : n \in \omega\} \subseteq \tau(X)$ with $\bigcap_{n=1}^\omega \text{cl}_X U_n = \{x\}$. Then $\bigcap_{n=1}^\omega \text{cl}_Z U_n(n) = \bigcap_{n=1}^\omega (\text{cl}_X U_n \cup (U_n \times [n, \omega) \times \hat{X}) \cup o(U_n)) = \{x\}$. Finally, for points of $\hat{X} \subset Z$, consider the following: let $p \in \hat{X}$; then $p$ is a free open ultrafilter on $X$ and $\bigcap_{U \in p} \text{cl}_X U = \emptyset$. But $X$ is also weakly real-compact; hence, there is a countable family $\mathcal{C} \subseteq p$ with $\bigcap_{U \in \mathcal{C}} \text{cl}_X U = \emptyset$. Considering the family $\mathcal{C}' = \{U(p) : U \in \mathcal{C}\}$, we have $\bigcap_{U \in \mathcal{C}'} \text{cl}_Z U(p) = \bigcap_{U \in \mathcal{C}} \text{cl}_X U \cup (\bigcap_{U \in \mathcal{C}} U \times \omega \times \{p\}) \cup \{p\} = \{p\}$. Hence, $\{p\}$ is the intersection of a countable collection of closed neighborhoods and $Z$ has countable closed-pseudocharacter. \qed
Now we construct a space, again modifying a construction of Bella and Yaschenko in [4], which is Urysohn, has countable closed-pseudocharacter, and has a large H-set.

**Theorem 5.** There is a space $Z$ with the following properties:

1. $Z$ is Urysohn;
2. $Z$ has countable closed-pseudocharacter;
3. $Z$ has an H-set $H$ of cardinality greater than $2^{\omega}$.

**Construction 6.** Let $\{X_n : n \in \omega\}$ be a sequence of spaces defined inductively as follows: $X_0 = \omega$ and $X_{n+1} = \hat{X}_n = \kappa X_n \setminus X_n$. For each $n \in \omega$, let $Z_n = X_n \cup (X_n \times \omega \times \hat{X}_n) \cup \hat{X}_n$. Finally, let $Z_\omega$ be the quotient space formed from $\bigcup_\omega Z_n$ by identifying $\hat{X}_{n-1}$ with $X_n$, and let $Z = Z_\omega \cup \{\infty\}$. A basic neighborhood of $\infty$ in $Z$ will be $\{\infty\} \cup \bigcup_{i \in \omega \setminus n} (X_i \times \omega \times \hat{X}_i : n \in \omega)$.

**Fact 7.** (1) The space $Z$ is Urysohn.
(2) $Z$ has countable closed-pseudocharacter.
(3) The set $H = \bigcup_\omega X_n \cup \{\infty\}$ is an H-set of $Z$.
(4) The cardinality of $H$ is larger than $2^{\phi(Z)} = \mathfrak{c}$.

**Proof:** (1) That the points which are isolated can be separated from the other points of $Z$ with closed neighborhoods is clear. Now for $x, y \in H$, if $x, y \in Z_n$ for some $n \in \omega$, then, by Fact 4(1), the two points can be separated by closed neighborhoods. If, on the other hand, $x \in Z_n$ and $y \in Z_m$ where $n \neq m$, then we may assume further that $x \in X_n$, $y \in X_m$, and $n < n + 2 \leq m$. Now let $U_x$ be a basic open set of $x$ and let $U_y$ be a basic open set of $y$. Then
\[
\text{cl}_Z U_x \subseteq X_{n-1} \cup (X_{n-1} \times \omega \times X_n) \cup X_n \cup (X_n \times \omega \times X_{n+1})
\]
while
\[
\text{cl}_Z U_y \subseteq X_{m-1} \cup (X_{m-1} \times \omega \times X_m) \cup X_m \cup (X_m \times \omega \times X_{m+1}).
\]
Hence, $\text{cl}_Z U_x \cap \text{cl}_Z U_y = \emptyset$.

Finally, if $x = \infty$ and $y \in Z_n$ for some $n$, we may simply take the basic open neighborhood $\{\infty\} \cup \bigcup_{i=n+2}^{\infty} (X_i \times \omega \times \hat{X}_i)$ for $x = \infty$ and a typical neighborhood for $y$.

(2) This is clear for each point in $Z_n$ for some $n$. For $\infty$, we let $\mathcal{U}$ be the neighborhood base $\{\infty\} \cup \bigcup_{i \in \omega \setminus n} (X_i \times \omega \times \hat{X}_i : n \in \omega)$. Then $\bigcap_{U \in \mathcal{U}} \text{cl}_Z U = \{\infty\}$.
Let $\mathcal{U}$ be a cover of $H$ with basic open sets of $Z$. There is some $U \in \mathcal{U}$ for which $\infty \in U$. Then for some $m \in \omega$, $\text{cl}_Z U$ contains $X_i$ for all $i \geq m$. The cover $\mathcal{U}$ contains, for each $n < m$, a subfamily $\mathcal{U}_n$ which covers $X_n \cup X_{n+1}$. Now, as in Fact 4(3) above, we have $H(X_n; X_n \cup X_{n+1}; Z_n)$. Hence, we obtain a finite subfamily $\mathcal{F}_n$ of $\mathcal{U}_n$, therefore of $\mathcal{U}$, for which $X_n \subseteq \bigcup \{U \in \mathcal{F}_n : n < m\}$. The collection $\{U\} \cup \bigcup \{\mathcal{F}_n : n < m\}$ is a finite family whose closure contains $H$.

(4) This is clear since $|X_1| = |\kappa \omega| = 2^\kappa \geq c$. □

**Theorem 8.** Let $X$ be a space, $\kappa = \bar{\psi}(X)$, $A$ an $H$-set of $X$, $K$ a compact Hausdorff space, and $f : K \to X$ a $\theta$-continuous function with $f[K] = A$; then $|A| \leq 2^\kappa$.

**Proof:** The proof follows the outline of the comments after Theorem 5 in [4]. Let $x \in A$ and take a family of open neighborhoods of $x$, $\{U_\alpha : \alpha \leq \kappa\}$, such that $\bigcap_\kappa \text{cl}_X U_\alpha = \{x\}$. For every $p \in f^- (x)$, fix an open neighborhood $W_{\alpha,p}$ satisfying $f[\text{cl}_K W_{\alpha,p}] \subseteq \text{cl}_X U_\alpha$ and let $W_\alpha = \bigcup \{W_{\alpha,p} : p \in f^- (x)\}$. Then $f[W_\alpha] \subseteq \text{cl}_X U_\alpha$ and hence, $f^- (x) \subseteq \bigcap_\kappa W_\alpha \subseteq f^- \bigcap_\kappa \text{cl}_X U_\alpha = f^- (x)$. Now Theorem 4 of [1] implies the family $\{f^- (x) : x \in A\}$ must have cardinality not more than $2^\kappa$, and hence $|A| \leq 2^\kappa$. □

**Corollary 9.** There is no compact Hausdorff space $K$ and $\theta$-continuous function $f : K \to Z$ with $f[K] = H$ for the space $Z$ and embedded $H$-set $H$ in Construction 6 above.

What is most notable about the above corollary is that the space $Z$ is a Urysohn counterexample to Vermeer’s conjecture. If the base space $X_0$ in the construction above is discrete, a few other properties of the space $Z$ are noteworthy. In this case, $Z$ is a simple Urysohn extension of the discrete space $\bigcup_\omega (X_i \times \omega \times \check{X}_i) = Z \setminus H$. One may also say that $Z$ is the disjoint union of discrete spaces, namely $Z = H \cup (Z \setminus H)$.

**Theorem 10.** If $A$ is an $H$-set of a Urysohn space $X$, then $|A| \leq 2^{\chi(X_s)}$.

**Proof:** Let $A$ be an $H$-set in a Urysohn space $X$. In [2], Bella shows $|A| \leq 2^{\chi(X)}$. Now $A \subset X_s$ is also an $H$-set of $X_s$, and $X_s$ is Urysohn. Hence, $|A| \leq 2^{\chi(X_s)}$. □
Acknowledgments. The author wishes to express thanks to Dr. Jack Porter for his encouragement and guidance during the preparation of this paper.

The useful suggestions of the referee are appreciated.

References


Department of Mathematics; University of Kansas; Lawrence, KS 66045
E-mail address: dmcneill@math.ku.edu