Topological Homeomorphism Groups
and Semi-Box Product Spaces

by

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Abstract. The space $\mathcal{H}_f(Y)$ of homeomorphisms on metric space $Y$ under the fine topology is shown to be a topological group. The space $\mathcal{H}_f^+(\mathbb{R})$ of increasing homeomorphisms on $\mathbb{R}$ has topological properties much like those of the box product $\Box \mathbb{R}^\omega$, but these two spaces are actually not homeomorphic. Under the motivation of finding a product space that is homeomorphic to $\mathcal{H}_f^+(\mathbb{R})$, the semi-box product $\triangledown \mathbb{R}^\omega$ is introduced, and its topological properties are studied. Among other relationships between the two spaces $\mathcal{H}_f^+(\mathbb{R})$ and $\triangledown \mathbb{R}^\omega$, is the one that each can be embedded in the other.

1. Introduction

For a Hausdorff space $Y$, let $\mathcal{H}(Y)$ be the group of (self) homeomorphisms on $Y$. If $\mathcal{H}_k(Y)$ denotes this group along with the compact-open topology, then this forms a topological group provided that $Y$ is either compact or locally compact locally connected [2]. But $\mathcal{H}_k(Y)$ may not be a topological group if $Y$ is a locally compact separable metric space, since the taking of inverses may not be a continuous operation, even for such a nice space $Y$. 

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In section 3, we show that if \( Y \) is a metric space and \( \mathcal{H}_f(Y) \) denotes \( \mathcal{H}(Y) \) with the fine topology, then \( \mathcal{H}_f(Y) \) is always a topological group. This is done by first showing that the fine topology on \( \mathcal{H}(Y) \) is equal to the graph topology there. We also look at some properties of \( \mathcal{H}_f(Y) \) by examining the equivalence classes of two natural equivalence relations on this space.

Turning now to specific spaces, let \( \mathbb{R} \) be the space of real numbers, let \( \mathbb{I} \) be the closed interval \([-1, 1]\), let \( \omega \) be the first infinite ordinal, and let \( \mathbb{N} \) be the set of natural numbers \( \omega \setminus \{0\} \). Let \( \mathcal{H}^+(\mathbb{R}) \) and \( \mathcal{H}^+(\mathbb{I}) \) be the increasing homeomorphisms in \( \mathcal{H}(\mathbb{R}) \) and \( \mathcal{H}(\mathbb{I}) \), respectively. It is evident that \( \mathcal{H}_k(\mathbb{R}) \) and \( \mathcal{H}_k(\mathbb{I}) \) are homeomorphic to the topological sum of two copies of \( \mathcal{H}_k^+(\mathbb{R}) \) and \( \mathcal{H}_k^+(\mathbb{I}) \), respectively. The same is true for \( \mathcal{H}_f(\mathbb{R}) \) and \( \mathcal{H}_f(\mathbb{I}) \). So to examine the properties of \( \mathcal{H}(\mathbb{R}) \) and \( \mathcal{H}(\mathbb{I}) \), we look only at \( \mathcal{H}^+(\mathbb{R}) \) and \( \mathcal{H}^+(\mathbb{I}) \).

In the late 1960s, R. D. Anderson, in an unpublished manuscript (Spaces of homeomorphisms of finite graphs), using techniques that are now called infinite-dimensional topology, proved that \( \mathcal{H}_k^+(\mathbb{I}) \) is homeomorphic to \( \mathbb{R}^\omega \), the product of \( \omega \) copies of \( \mathbb{R} \) with the Tychonoff product topology (see also [5] and [13]). The space \( \mathcal{H}^+_k(\mathbb{R}) \) is naturally homeomorphic to \( \mathcal{H}_k^+(\mathbb{I}) \), and is thus homeomorphic to \( \mathbb{R}^\omega \). Also \( \mathcal{H}^+_f(\mathbb{I}) \) is equal to \( \mathcal{H}_k^+(\mathbb{I}) \), so it too is homeomorphic to \( \mathbb{R}^\omega \). However, \( \mathcal{H}^+_f(\mathbb{R}) \) has a strictly finer topology than that of \( \mathcal{H}_k^+(\mathbb{R}) \), so the question arises as to whether the topological group \( \mathcal{H}^+_f(\mathbb{R}) \) is homeomorphic to \( \mathbb{R}^\omega \) with some naturally defined topology that is finer than the Tychonoff product topology—perhaps the box product topology.

In section 4, we show that the topological properties of \( \mathcal{H}^+_f(\mathbb{R}) \) are similar to those of \( \square \mathbb{R}^\omega \), the space \( \mathbb{R}^\omega \) with the box product topology. However, we end up showing that \( \mathcal{H}^+_f(\mathbb{R}) \) is not actually homeomorphic to \( \square \mathbb{R}^\omega \).

Then in section 5, we introduce what we call the semi-box product topology, which is finer than the Tychonoff product topology and coarser than the box product topology. This semi-box product topology on \( \mathbb{R}^\omega \) gives a space, denoted by \( \diamondsuit \mathbb{R}^\omega \), that seems to be a good candidate for a space homeomorphic to \( \mathcal{H}^+_f(\mathbb{R}) \).

Finally, in section 6, we study the properties of \( \diamondsuit \mathbb{R}^\omega \) and give some results that support the conjecture that \( \mathcal{H}^+_f(\mathbb{R}) \) is homeomorphic to \( \diamondsuit \mathbb{R}^\omega \). In particular, we show that \( \mathcal{H}^+_f(\mathbb{R}) \) and \( \diamondsuit \mathbb{R}^\omega \)
can each be embedded in the other. Also $H_f^+(\mathbb{R})$ is shown to be homeomorphic to $\square Q \times \square \mathbb{R}^\omega$ where $\square Q$ is a certain subspace of $\square \mathbb{R}^\omega$.

2. Properties of $H_k(Y)$

The main theorem concerning the group $H_k(Y)$ of homeomorphisms on a Hausdorff space $Y$, where the function space topology is the compact-open topology, is the following theorem due to Richard F. Arens in [2].

**Theorem 2.1.** If $Y$ is either compact or locally compact locally connected, then $H_k(Y)$ is a topological group.

This was improved by Jan J. Dijkstra in [7] where it is shown that if $Y$ is a Hausdorff space such that every point has a neighborhood that is a continuum, then $H_k(Y)$ is a topological group. In [7] an example is given showing that if $Y$ is the Cantor set minus a point, then the inverse operator on $H_k(Y)$ is not continuous, showing that $H_k(Y)$ does not have to be a topological group even if $Y$ is a locally compact separable metric space.

Since $H_k(Y)$ is a subspace of the space $C_k(Y)$ of continuous real-valued functions on $Y$, with the compact-open topology, $H_k(Y)$ inherits certain topological properties from $C_k(Y)$. In particular, if $Y$ is a locally compact separable metric space, then $C_k(Y)$ is a separable Banach space (see [3], [14], [21], [22]), so that if $Y$ is infinite, $C_k(Y)$ is homeomorphic to the product space $\mathbb{R}^\omega$ with the Tychonoff product topology (see [4], [20]). Therefore, for locally compact separable metric spaces $Y$, $H_k(Y)$ can be embedded into $\mathbb{R}^\omega$.

Now let us consider the topological groups $H_k^+(\mathbb{I})$ and $H_k^+(\mathbb{R})$, where the homeomorphisms are the ones that are increasing functions. As indicated in the Introduction, we have the following theorem of Anderson.

**Theorem 2.2.** The topological group $H_k^+(\mathbb{I})$ is homeomorphic to $\mathbb{R}^\omega$.

We show that $H_k^+(\mathbb{R})$ is also homeomorphic to $\mathbb{R}^\omega$ by showing that there is a natural homeomorphism from $H_k^+(\mathbb{I})$ onto $H_k^+(\mathbb{R})$. This is well known, but we want to give the argument using the following lemma that is also used in section 5. In this lemma, $C_k^+(\mathbb{I})$
is the space of strictly increasing continuous real-valued functions on \( \mathbb{I} \). This function space has the compact-open topology, which for a general space \( C_k(Y) \) has a base consisting of sets of the form

\[
B(f, K, \varepsilon) = \{ g \in C_k(Y) : |f(t) - g(t)| < \varepsilon \text{ for all } t \in K \}
\]

where \( f \in C_k(Y) \), \( K \) is a compact subset of \( Y \), and \( \varepsilon > 0 \).

**Lemma 2.3.** If \( D \) is a dense subset of \( \mathbb{I} \), then \( C^+_k(\mathbb{I}) \) has a base consisting of sets of the form

\[
B(f, F, \varepsilon) = \{ g \in C^+_k(\mathbb{I}) : |f(t) - g(t)| < \varepsilon \text{ for all } t \in F \}
\]

where \( f \in C^+_k(\mathbb{I}) \), \( F \) is a finite subset of \( D \), and \( \varepsilon > 0 \). In particular, the compact-open topology is equal to the topology of pointwise convergence on \( C^+_k(\mathbb{I}) \).

**Proof:** Each \( B(f, F, \varepsilon) \) is clearly open in \( C^+_k(\mathbb{I}) \), so let \( f \in C^+_k(\mathbb{I}) \) and \( \varepsilon > 0 \). We need to find a finite subset \( F \) of \( D \) such that \( B(f, F, \varepsilon/12) \subseteq B(f, \mathbb{I}, \varepsilon) \).

For each \( t \in \mathbb{I} \), let \( U(t) \) be an open interval intersected with \( \mathbb{I} \) such that

\[
f(U(t)) \subseteq (h(t) - \varepsilon/12, h(t) + \varepsilon/12).
\]

By the compactness of \( \mathbb{I} \), there exist \(-1 = t_1 < t_2 < \cdots < t_{m-1} < t_m = 1\) in \( \mathbb{I} \) such that

\[
\mathbb{I} = U(t_1) \cup \cdots \cup U(t_m).
\]

By choosing subsets if necessary, we may assume that each \( t_i \) is not in \( \overline{U(t_j)} \) for any \( j \neq i \). Then for each \( i = 1, \ldots, m \), define

\[
U_i = U(t_i) \setminus \cup \{ \overline{U(t_j)} : j = 1, \ldots, m \text{ and } j \neq i \}.
\]

For each \( i = 1, \ldots, m \), let \( d_i \in U_i \cap D \), and define \( F = \{ d_1, \ldots, d_m \} \).

To show that \( B(f, F, \varepsilon/12) \subseteq B(f, \mathbb{I}, \varepsilon) \), let \( g \in B(f, F, \varepsilon/12) \) and let \( t \in \mathbb{I} \). Then \( t \in U_i \) for some \( i = 1, \ldots, m \). We consider only the case that \( 1 < i < m \) since the cases that \( i = 1 \) and \( i = m \) are similar.

Now \( |g(d_i) - f(d_i)| < \varepsilon/12 \). Also \( |f(d_i) - f(t_i)| < \varepsilon/12 \) and \( |f(t) - f(t_i)| < \varepsilon/12 \). From the first two inequalities, we have \( |g(d_i) - f(t_i)| < \varepsilon/6 \). From this and the third inequality, we have \( |g(d_i) - f(t)| < \varepsilon/4 \).

Note that \( U(t_{i-1}) \cap U(t_i) \neq \emptyset \), so that

\[
(f(t_{i-1}) - \varepsilon/12, f(t_{i-1}) + \varepsilon/12) \cap (f(t_i) - \varepsilon/12, f(t_i) + \varepsilon/12) \neq \emptyset.
\]
From this we obtain the fact that \( f(t_i) - \varepsilon / 12 < f(t_{i-1}) + \varepsilon / 12 \), so that \( f(t_i) - f(t_{i-1}) < \varepsilon / 6 \). Arguing as above, we have \( |g(d_{i-1}) - f(t_{i-1})| < \varepsilon / 6 \), so that \( |g(d_{i-1}) - f(t_i)| < \varepsilon / 3 \). Now we can conclude that \( |g(d_{i-1}) - g(d_i)| < \varepsilon / 2 \).

Similarly, we can argue that \( |g(d_{i+1}) - g(d_i)| < \varepsilon / 2 \). Now either \( d_{i-1} < t \leq d_i \) or \( d_i \leq t < d_{i+1} \), so that either \( g(d_{i-1}) < g(t) \leq g(d_i) \) or \( g(d_i) \leq g(t) < g(d_{i+1}) \). Therefore, \( |g(t) - g(d_i)| < \varepsilon / 2 \). But since \( |g(d_i) - f(t)| < \varepsilon / 4 \), we have \( |g(t) - f(t)| < 3\varepsilon / 4 < \varepsilon \). It follows that \( g \in B(f, I, \varepsilon) \), and thus \( B(f, F, \varepsilon/12) \subseteq B(f, I, \varepsilon) \).

**Theorem 2.4.** There is a natural topological group isomorphism from \( \mathcal{H}_k^+(I) \) onto \( \mathcal{H}_k^+(\mathbb{R}) \).

**Proof:** Let \( \tau : (-1, 1) \rightarrow \mathbb{R} \) be the homeomorphism defined by

\[
\tau(t) = \tan\left(\frac{\pi t}{2}\right)
\]

for all \( t \in (-1, 1) \). Define \( \eta : \mathcal{H}_k^+(I) \rightarrow \mathcal{H}_k^+(\mathbb{R}) \) by

\[
\eta(h) = \tau h \tau^{-1}
\]

for all \( h \in \mathcal{H}_k^+(I) \). Note that \( \tau^{-1} \) is defined by \( \eta^{-1}(g) = \tau^{-1} g \tau \). Now \( \eta \) is clearly a group isomorphism, so we show that it is also a homeomorphism.

Let \( h \in \mathcal{H}_k^+(I) \) and let \( B(\eta(h), K, \varepsilon) \) be a basic neighborhood of \( \eta(h) \) in \( \mathcal{H}_k^+(\mathbb{R}) \) where \( K \) is a compact subset of \( \mathbb{R} \). By the continuity of \( \tau^{-1} \), \( \tau^{-1}(K) \) is a compact subset in \( (-1, 1) \), and hence compact in \( I \). Therefore, by the continuity of \( \tau \), there exists a \( \delta > 0 \) such that for each \( r, s \in \tau^{-1}(K) \) with \( |r - s| < \delta \), \( |\tau(r) - \tau(s)| < \varepsilon \). Then, if \( f \in B(h, \tau^{-1}(K), \delta) \), for every \( t \in K \),

\[
|f(\tau^{-1}(t)) - h(\tau^{-1}(t))| < \delta,
\]

so that

\[
|\eta(f)(t) - \eta(h)(t)| = |\tau f(\tau^{-1}(t)) - \tau h(\tau^{-1}(t))| < \varepsilon.
\]

This shows that

\[
\eta(B(h, \tau^{-1}(K), \delta)) \subseteq B(\eta(h), K, \varepsilon),
\]

and hence \( \eta \) is continuous.

To show that \( \eta^{-1} \) is continuous, let \( h \in \mathcal{H}_k^+(\mathbb{R}) \), and let \( B(\eta^{-1}(h), F, \varepsilon) \) be a basic neighborhood of \( \eta^{-1}(h) \) in \( \mathcal{H}_k^+(I) \) as given
by Lemma 2.3, where $F$ is a finite subset of $(-1,1)$. Define 

$$K = \{\tau(t) : t \in F\}.$$ 

Then $B(h, K, \varepsilon)$ is a neighborhood of $h$ in $\mathcal{H}_k^+(\mathbb{R})$. Since the derivative of $\tau^{-1}$ is less than 1 at all points of $\mathbb{R}$, we have that for every $r, s \in \mathbb{R}$, $|\tau^{-1}(r) - \tau^{-1}(s)| \leq |r - s|$. So if $f \in B(h, K, \varepsilon)$, for every $t \in F$,

$$|\eta^{-1}(f)(t) - \eta^{-1}(h)(t)| = |\tau^{-1}f(\tau(t)) - \tau^{-1}h(\tau(t))| 
\leq |f(\tau(t)) - h(\tau(t))| < \varepsilon.$$

This shows that $\eta^{-1}(f) \in B(\eta^{-1}(h), F, \varepsilon)$, and thus

$$\eta^{-1}(B(h, K, \varepsilon)) \subseteq B(\eta^{-1}(h), F, \varepsilon).$$

So $\eta^{-1}$ is continuous, and it now follows that $\eta$ is a homeomorphism.

Since $\mathcal{H}_k^+(\mathbb{R})$ is homeomorphic to $\mathcal{H}_k^+(\mathbb{I})$ by Theorem 2.4, we see that $\mathcal{H}_k^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega$ by Theorem 2.2. Now $\mathbb{I}$ is compact, so that $\mathcal{H}_f^+(\mathbb{I})$ with the fine topology is equal to $\mathcal{H}_k^+(\mathbb{I})$, and hence $\mathcal{H}_f^+(\mathbb{I})$ is also homeomorphic to $\mathbb{R}^\omega$. But the fine topology on $\mathcal{H}_f^+(\mathbb{R})$ is strictly finer than the compact-open topology on $\mathcal{H}_k^+(\mathbb{R})$, so we are interested in understanding just what topological space $\mathcal{H}_f^+(\mathbb{R})$ is. In the next section we show that for general metric spaces $Y$, the space $\mathcal{H}_f(Y)$ is a topological group, and we examine some topological properties of this space.

3. Properties of $\mathcal{H}_f(Y)$

Let $C(X, Y)$ be the set of continuous functions from the topological space $X$ into the topological space $Y$. If $Y$ is a metric space with metric $d$, then the fine topology on $C(X, Y)$ (with respect to $d$) has a base consisting of sets of the form

$$B(f, \varepsilon) = \{g \in C(X, Y) : \text{for all } x \in X, \ d(f(x), g(x)) < \varepsilon(x)\}$$

where $f \in C(X, Y)$ and $\varepsilon \in C_+(X)$, the set of positive continuous real-valued functions on $X$. The fine topology is also called the Whitney topology, the Morse topology, and the $m$-topology (see [8], [10], [15]).

If $X$ is a binormal space (that is, a countably paracompact normal space), then the fine topology on $C(X, Y)$ turns out to be
independent of the metric $d$ on $Y$, because in this case, such a topology is equal to the graph topology on $C(X,Y)$ having a base consisting of sets of the form

$$W^+ = \{ f \in C(X,Y) : f \subseteq W \}$$

where $W$ is an open subset of $X \times Y$ and each function in $C(X,Y)$ is identified with its graph (see [16], [17]).

**Theorem 3.1.** If $X$ is a binormal space and $Y$ is a metric space, then the fine topology on $C(X,Y)$ is equal to the graph topology on $C(X,Y)$.

**Proof:** Let $f \in C(X,Y)$ and let $\varepsilon \in C_+(X)$. To show that $B(f, \varepsilon)$ is open in the graph topology, define

$$W = \bigcup \{ \{ x \} \times B(f(x), \varepsilon(x)) : x \in X \}$$

where $B(f(x), \varepsilon(x))$ is the open ball in $Y$ centered at $f(x)$ and having radius $\varepsilon(x)$. We need to show that $W$ is open in $X \times Y$, so let $(x, y) \in W$. Then $d(y, f(x)) < \varepsilon(x)$, so we can define positive number $\delta = \varepsilon(x) - d(y, f(x))$. By the continuity of $f$ and $\varepsilon$, $x$ has a neighborhood $U$ such that $f(U) \subseteq B(f(x), \delta/3)$ and $\varepsilon(U)$ is contained in the open interval $(\varepsilon(x) - \delta/3, \varepsilon(x) + \delta/3)$.

To show that $U \times B(y, \delta/3) \subseteq W$, let $x' \in U$ and $y' \in B(y, \delta/3)$. Then

$$d(y', f(x')) \leq d(y', y) + d(y, f(x)) + d(f(x), f(x'))$$

$$< \delta/3 + d(y, f(x)) + \delta/3$$

$$= \delta/3 + \varepsilon(x) - \delta + \delta/3$$

$$= \varepsilon(x) - \delta/3$$

$$< \varepsilon(x').$$

Hence, $(x', y') \in \{ x' \} \times B(f(x'), \varepsilon(x')) \subseteq W$. Therefore, $U \times B(y, \delta/3)$ is a neighborhood of $(x, y)$ contained in $W$, showing that $W$ is open in $X \times Y$. One can see that $B(f, \varepsilon) = W^+$, and thus $B(f, \varepsilon)$ is open in the graph topology on $C(X,Y)$.

Now suppose that $X$ is a binormal space and let $W$ be an open subset of $X \times Y$. To show that $W^+$ is open in the fine topology, let $f \in W^+$. We need to find an $\varepsilon \in C_+(X)$ such that $B(f, \varepsilon) \subseteq W^+$.

For each $x \in X$, there exist a neighborhood $U_x$ of $x$ and an element $n_x$ of $\mathbb{N}$ such that $U_x \times B(f(x), 1/n_x) \subseteq W$. Because of the
continuity of $f$, we can take $U_x$ so that $f(U_x) \subseteq B(f(x), 1/(2n_x))$.
For each $m \in \mathbb{N}$, let

$$U_m = \bigcup \{U_x : x \in X \text{ and } n_x = m \}.$$ 

Since $X$ is countably paracompact, the countable open cover $\{U_m : m \in \mathbb{N}\}$ of $X$ has a locally finite refinement $\mathcal{U}$. For each $U \in \mathcal{U}$, let $m_U \in \mathbb{N}$ be such that $U \subseteq U_{m_U}$. Now define $\delta : X \to (0, \infty)$ by

$$\delta(x) = \min \{1/m_U : U \in \mathcal{U} \text{ and } x \in U \}$$

for all $x \in X$.

To show that $\delta$ is lower semicontinuous, let $x \in X$. Now $x$ has a neighborhood $U'$ that intersects only finitely many members of $\mathcal{U}$, say $U_1, \ldots, U_k$. We may assume that $x \in U_1 \cap \cdots \cap U_k$ because if $x \notin U_i$, then we can use $U' \setminus U_i$ as a neighborhood of $x$. Then we have $\delta(x') \geq \delta(x)$ for all $x' \in U'$, showing that $\delta$ is lower semicontinuous. Since $X$ is binormal and $\delta$ is positive, there exists an $\varepsilon \in C_+(X)$ such that $\varepsilon < \delta$.

To show that $B(f, \varepsilon/2) \subseteq W^+$, let $g \in B(f, \varepsilon/2)$ and let $x \in X$. Then $g(x) \in B(f(x), \varepsilon(x)/2)$. There is some $U \in \mathcal{U}$ with $x \in U$, so that $\varepsilon(x) < \delta(x) \leq 1/m_U$. Now $U \subseteq U_{m_U}$ and $m_U = n_{x_0}$ for some $x_0 \in X$ with $x \in U_{x_0}$. Also $U_{x_0} \times B(f(x_0), 1/n_{x_0}) \subseteq W$. Since $f(U_{x_0}) \subseteq B(f(x_0), 1/(2n_{x_0}))$, we have

$$d(g(x), f(x_0)) \leq d(g(x), f(x)) + d(f(x), f(x_0))$$

$$< \varepsilon(x)/2 + 1/(2n_{x_0})$$

$$< 1/n_{x_0}.$$ 

Therefore, $\langle x, g(x) \rangle \in U_{x_0} \times B(f(x_0), 1/n_{x_0}) \subseteq W$, and we have $g \in W^+$. This shows that $W^+$ is open in the fine topology on $C(X, Y)$, so that the fine and graph topologies on $C(X, Y)$ are equal. □

For a metric space $Y$, the space $\mathcal{H}_f(Y)$ of homeomorphisms on $Y$ is a subspace of $C(Y, Y)$ with the fine topology. In this case, Theorem 3.1 says that the topology on $\mathcal{H}_f(Y)$ is also equal to the graph topology.

**Theorem 3.2.** If $Y$ is a metric space, then $\mathcal{H}_f(Y)$ with the fine topology is a topological group.

**Proof:** The proof of the continuity of inversion in $\mathcal{H}_f(Y)$ is easy using the graph topology. Let $f \in \mathcal{H}_f(Y)$ and let $W$ be an open
subset of \( Y \times Y \) with \( f^{-1} \in W^+ \). Then if \( W^{-1} = \{ \langle x, y \rangle : \langle y, x \rangle \in W \} \), it is clear that \( W^{-1} \) is open in \( Y \times Y \) and that \( f \in (W^{-1})^+ \). But if \( g \in (W^{-1})^+ \), then \( g^{-1} \in W^+ \), which shows that the inverse in \( \mathcal{H}_f(Y) \) is a continuous operation.

The proof of the continuity of composition in \( \mathcal{H}_f(Y) \) is not so easy using the graph topology. We need to use the metric structure on \( Y \), and so we use the fine topology on \( \mathcal{H}_f(Y) \).

Let \( f, g \in \mathcal{H}_f(Y) \) and \( \varepsilon \in C_+(Y) \). Note that \( \varepsilon f^{-1} \in C_+(Y) \). Now let us define \( \delta \in \mathcal{H} \), and so we use the fine topology on \( \mathcal{H}_f(Y) \).

Now let us define \( \delta : Y \to (0, \infty) \) by

\[
\delta(y) = \sup \{ r \in (0, \infty) : \text{for some } s \in (0, \infty), \quad g(B(y, r)) \subseteq B(g(y), \varepsilon f^{-1}(y) - s),
\]

\[
\text{and } \varepsilon f^{-1}(B(y, r)) \subseteq (s, 2\varepsilon f^{-1}(y) - s) \}
\]

for all \( y \in Y \).

To show that \( \delta \) is lower semicontinuous, let \( y \in Y \) and \( a \in (0, \infty) \). Then there exist \( r, s \in (0, \infty) \) such that \( r > \delta(y) - a, g(B(y, r)) \subseteq B(g(y), \varepsilon f^{-1}(y) - s) \), and \( \varepsilon f^{-1}(B(y, r)) \subseteq (s, 2\varepsilon f^{-1}(y) - s) \). Let \( t = (r - \delta(y) + a)/2 \). Finally, define

\[
U = B(y, t) \cap g^{-1}(B(g(y), s/3)) \cap f^{-1}(\varepsilon f^{-1}(y) - s/3, \varepsilon f^{-1}(y) + s/3),
\]

which is a neighborhood of \( y \) in \( Y \).

We need to show that \( \delta(U) \subseteq (\delta(x) - a, \infty) \). So let \( y' \in U \). Now take \( r' = r - t \), so that \( \delta(y) - a < r' < r \). Observe that \( B(y', r') \subseteq B(y, r) \) because \( y' \in B(y, t) \) and \( r' + t = r \). So we have

\[
g(B(y', r')) \subseteq B(g(y), \varepsilon f^{-1}(y) - s)
\]

and

\[
\varepsilon f^{-1}(B(y', r')) \subseteq (s, 2\varepsilon f^{-1}(y) - s).
\]

We also have \( g(y') \in B(g(y), s/3) \) and \( \varepsilon f^{-1}(y^{-1}) \in (\varepsilon f^{-1}(y) - s/3, \varepsilon f^{-1}(y) + s/3) \).

Now we need to show that

\[
B(g(y), \varepsilon f^{-1}(y) - s) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3).
\]

So let \( z \in B(g(y), \varepsilon f^{-1}(y) - s) \). Then

\[
d(z, g(y')) \leq d(z, g(y)) + d(g(y), g(y'))
\]

\[
< \varepsilon f^{-1}(y) - s + s/3
\]

\[
< (\varepsilon f^{-1}(y') + s/3) - s + s/3
\]

\[
= \varepsilon f^{-1}(y') - s/3.
\]
Therefore,
\[ B(g(y), \varepsilon f^{-1}(y) - s) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3), \]
showing that
\[ g(B(y', r')) \subseteq B(g(y'), \varepsilon f^{-1}(y') - s/3). \]

With the same argument as above, we also get that
\[ (s, \varepsilon f^{-1}(y) - s) \subseteq (s/3, \varepsilon f^{-1}(y') - s/3), \]
which shows that
\[ \varepsilon f^{-1}(B(y', r')) \subseteq (s/3, \varepsilon f^{-1}(y') - s/3). \]

We can now conclude that \( r' \leq \delta(y') \), and hence \( \delta(y) - a < r' \leq \delta(y') \). This is true for all \( y' \in U \), so that \( \delta(U) \subseteq (\delta(y) - a, \infty) \), and thus \( \delta \) is lower semicontinuous.

Since \( 0 < \delta \), there is a \( \sigma \in C_{+}(Y) \) such that \( \sigma < \delta \). Note that \( \sigma f \in C_{+}(Y) \). Consider the neighborhoods \( B(f, \sigma f) \) and \( B(g, \varepsilon f^{-1}) \) of \( f \) and \( g \) in \( H_f(Y) \). We want to show that if \( f' \in B(f, \sigma f) \) and \( g' \in B(g, \varepsilon f^{-1}) \), then \( g' f' \in B(gf, 3\varepsilon) \) (by using \( \varepsilon/3 \) in defining \( \delta \) and by taking \( g' \) from \( B(g, \varepsilon f^{-1}/3) \), we can get \( g' f' \in B(gf, \varepsilon) \)).

So to show that \( g' f' \in B(gf, 3\varepsilon) \), let \( y \in Y \). Then \( f'(y) \in B(f(y), \sigma f(y)) \). Now \( \sigma f(y) < \delta(f(y)) \), so that
\[
\begin{align*}
g(B(f(y), \sigma f(y))) & \subseteq B(g(f(y)), \varepsilon f^{-1}(f(y))) \\
& = B(gf(y), \varepsilon(y)).
\end{align*}
\]
Therefore, \( gf'(y) \in B(gf(y), \varepsilon(y)) \). Also since \( g' \in B(g, \varepsilon f^{-1}) \), we have \( g' f'(y) \in B(gf'(y), \varepsilon f^{-1}(f'(y))) \). But
\[
\varepsilon f^{-1}(B(f(y), \sigma f(y))) \subseteq (0, 2\varepsilon(y)),
\]
so that \( \varepsilon f^{-1}(f'(y)) \in (0, 2\varepsilon(y)) \); that is, \( \varepsilon f^{-1}(f'(y)) < 2\varepsilon(y) \). So \( g' f'(y) \in B(gf'(y), 2\varepsilon(y)) \), and thus \( g' f'(y) \in B(gf(y), 3\varepsilon(y)) \), as needed. This finishes the argument that composition in \( H_f(Y) \) is continuous.

We now try to get some understanding of the structure of the space \( H_f(Y) \) by looking at the equivalence classes of two natural equivalence relations defined on \( H(Y) \). Let us begin by considering the more general space \( C(X, Y) \).

First, let \( \approx \) be the equivalence relation on \( C(X, Y) \) defined by \( f \approx g \) provided that there exists a compact subset \( K \) of \( X \) such that \( f(x) = g(x) \) for all \( x \in X \setminus K \). For each \( f \in C(X, Y) \), let
Proposition 3.3. If \( X \) is a locally compact \( \sigma \)-compact space and \( Y \) is a metric space, then \( E(f) \) is a closed subspace of \( C_f(X,Y) \) for all \( f \in C(X,Y) \).

Proof: Since this is obviously true for \( X \) compact, we assume that \( X \) is not compact. Then we can write \( X = \bigcup \{ K_n : n \in \mathbb{N} \} \) where each \( K_n \) is compact and contained in the interior of \( K_{n+1} \). Let \( f \in C(X,Y) \) and \( g \in C(X,Y) \setminus E(f) \). Then for each \( n \in \mathbb{N} \), there exists an \( x_n \in X \setminus K_n \) such that \( g(x_n) \neq f(x_n) \); let \( \varepsilon_n = d(g(x_n), f(x_n)) \). Now \( \{ x_n : n \in \mathbb{N} \} \) is a closed discrete subset of \( X \), so that the function from \( \{ x_n : n \in \mathbb{N} \} \) into \((0,\infty)\) mapping each \( x_n \) to \( \varepsilon_n \) has an extension to some \( \varepsilon \in C_+ (X) \). It is evident that \( B(g,\varepsilon) \subseteq C_f(X,Y) \setminus E(f) \), and this shows that \( E(f) \) is closed in \( C_f(X,Y) \).

Corollary 3.4. If \( Y \) is a locally compact separable metric space, then \( E(h) \) is a closed subspace of \( \mathcal{H}_f(Y) \) for all \( h \in \mathcal{H}(Y) \).

Let \( e \) denote the identity map in \( \mathcal{H}(Y) \).

Proposition 3.5. For every space \( Y \), \( E(e) \) is a normal subgroup of \( \mathcal{H}(Y) \).

Proof: Let \( f, g \in E(e) \). Then there are compact subsets \( K_1 \) and \( K_2 \) of \( Y \) such that \( f(y) = y \) for all \( y \in Y \setminus K_1 \) and \( g(y) = y \) for all \( y \in Y \setminus K_2 \). The set \( f^{-1}(K_2) \) is compact in \( Y \), so that the set \( K = K_1 \cup f^{-1}(K_2) \) is compact. If \( y \in Y \setminus K \), then \( f(y) \notin Y \setminus K_2 \), so that \( f(y) = y \) and \( g(f(y)) = f(y) = y \). Therefore, \( gf \in E(e) \). Also, \( f(K_1) \) is compact, and if \( y \in Y \setminus f(K_1) \), then \( f^{-1}(y) \notin Y \setminus K_1 \), so that \( y = f(f^{-1}(y)) = f^{-1}(y) \). It follows that \( f^{-1} \in E(e) \) and completes the argument that \( E(e) \) is a subgroup of \( \mathcal{H}(Y) \).

To show that \( E(e) \) is a normal subgroup of \( \mathcal{H}(Y) \), let \( f \in E(e) \) and let \( g \in \mathcal{H}(Y) \). Then there exists a compact subset \( K \) of \( Y \) such that \( f(y) = y \) for all \( y \in Y \setminus K \). Let \( K' = g(K) \), which is a compact subset of \( Y \). Then, if \( y \in Y \setminus K' \), we have \( g^{-1}(y) \in Y \setminus K \), so that \( gfg^{-1}(y) = g(f(g^{-1}(y))) = g(g^{-1}(y)) = y \). Thus, \( gfg^{-1} \in E(e) \), showing that \( E(e) \) is indeed a normal subgroup of \( \mathcal{H}(Y) \).

Corollary 3.6. If \( Y \) is a locally compact separable metric space, then the quotient group \( \mathcal{H}_f(Y)/E(e) \) is a topological group under
the quotient topology, which implies that $E(h)$ is homeomorphic to $E(e)$ for all $h \in \mathcal{H}(Y)$.

**Example 3.7.** For $Y = \mathbb{R}$, the subgroup $E(e)$ of $\mathcal{H}_f(Y)$ is not open in $\mathcal{H}_f(Y)$, which implies that $\mathcal{H}_f(Y)/E(e)$ is not a discrete group. To show that $E(e)$ is not open, let $D_1(Y)$ be the set of $\delta \in C(Y)$ that $1$s are differentiable with $|\delta'(y)| < 1$ for all $y \in Y$. Let $\varepsilon \in C_+(Y)$. Then we can find a $\delta \in C_+(Y) \cap D_1(Y)$ such that $\delta < \varepsilon$. Letting $f = e + \delta$, we have $f$ strictly increasing so that it is in $\mathcal{H}(Y)$. Also $f \in B(e, \varepsilon)$. But $f(y) \neq y$ for all $y \in Y$, and hence $f \notin E(e)$. Since $\varepsilon$ is arbitrary, we see that $E(e)$ is not open.

For a second equivalence relation on $(C(X,Y))$, let us take $Y$ to be a metric space with metric $d$. Let $\sim$ be the equivalence relation on $C(X,Y)$ defined by $f \sim g$ provided that for every $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $d(f(x),g(x)) < \varepsilon$ for all $x \in X \setminus K$. For each $f \in C(X,Y)$, let $F(f)$ be the equivalence class of $\sim$ that contains $f$. It is clear that $E(f) \subseteq F(f)$ for all $f \in C(X,Y)$.

**Proposition 3.8.** If $X$ is any space and $Y$ is a metric space, then $F(f)$ is a closed subspace of $C_f(X,Y)$ for all $f \in C(X,Y)$. Furthermore, if $X$ is a locally compact $\sigma$-compact space, then $F(f)$ is an open subspace of $C_f(X,Y)$ for all $f \in C(X,Y)$, which implies that $C_f(X,Y)$ is equal to the topological sum of the distinct members of $\{F(f) : f \in C(X,Y)\}$.

**Proof:** To show that $F(f)$ is closed in $C_f(X,Y)$, let $g \in C_f(X,Y) \setminus F(f)$. Then there exists a $\delta > 0$ such that for every compact subset $K$ of $X$, there is an $x \in X \setminus K$ with $d(g(x),f(x)) \geq \delta$. Let $\varepsilon \in C_+(X)$ be the constant function on $X$ with value $\delta/2$. If $h \in B(g, \varepsilon)$, then for each compact subset $K$ of $X$, there exists an $x \in X \setminus K$ such that

$$\delta \leq d(g(x),f(x)) \leq d(g(x),h(x)) + d(h(x),f(x)),$$

and thus, $d(h(x),f(x)) < \delta/2 + d(h(x),f(x))$, which shows that $h \notin F(f)$, and hence $B(g, \varepsilon) \subseteq C_f(X,Y) \setminus F(f)$, finishing the proof that $F(f)$ is closed.

If $Y$ is a locally compact $\sigma$-compact space, we can write $X = \bigcup \{K_n : n \in \mathbb{N}\}$ where each $K_n$ is compact and contained in the
interior of $K_{n+1}$. To show that $F(f)$ is open in $C_f(X,Y)$, first choose an $\varepsilon \in C_+(X)$ such that for every $n \in \mathbb{N}$ and $x \in K_n$, $\varepsilon(x) < 1/n$. Now let $g \in F(f)$ and let $h \in B(g, \varepsilon)$. To show that $h \in F(f)$, let $\delta > 0$. Then take an $n \in \mathbb{N}$ with $1/n < \delta$ and let $x \in X \setminus K_n$. So we have $d(h(y), g(y)) < \varepsilon(x) < 1/n < \delta$, which shows that $h \sim g$. Since $g \sim f$, we have $h \sim f$, and thus $h \in F(f)$. Therefore, $B(g, \varepsilon) \subseteq F(f)$, and since $g$ is arbitrary, $F(f)$ is open in $C_f(X,Y)$.

Corollary 3.9. If $Y$ is a locally compact separable metric space, then $F(h)$ is an open and closed subspace of $\mathcal{H}_f(Y)$ for all $h \in \mathcal{H}(Y)$, which implies that $\mathcal{H}_f(Y)$ is equal to the topological sum of the distinct members of $\{F(h) : h \in \mathcal{H}(Y)\}$.

Example 3.10. If $Y = \mathbb{R}^\omega$, then $F(e)$ is the trivial subgroup $\{e\}$ in $\mathcal{H}(Y)$, which is not open in $\mathcal{H}(Y)$. This shows that the local compactness hypothesis in Corollary 3.9 cannot be dropped. To show that $F(e) = \{e\}$, let $f \in \mathcal{H}_f(Y) \setminus \{e\}$. Then there exists a $y_0 \in Y$ such that $f(y_0) \neq y_0$. Define $\delta = d(f(y_0), y_0)$. Then $y_0$ has a neighborhood $U$ in $Y$ such that $d(f(y), y) \geq \delta/2$ for all $y \in U$. For each compact subset $K$ of $Y$, there exists a $y \in U \setminus K$, and hence $d(f(y), y) \geq \delta/2$. This shows that $f \notin F(e)$, and therefore, $E(e) = F(e) = \{e\}$.

Proposition 3.11. For every space $Y$, $F(e)$ is a subgroup of $\mathcal{H}(Y)$.

Proof: This argument is similar to that in Proposition 3.5, except for the need to use $\varepsilon/2$ and the triangle inequality property of $d$ to show that $F(e)$ is closed under composition. □

Corollary 3.12. If $Y$ is a locally compact separable metric space, then $F(e)$ is an open and closed subgroup of the topological group $\mathcal{H}_f(Y)$.

Example 3.13. For $Y = \mathbb{R}$, the subgroup $F(e)$ of $\mathcal{H}_f(Y)$ is not a normal subgroup of $\mathcal{H}(Y)$. To show that $F(e)$ is not normal, let $f, g \in \mathcal{H}(Y)$ be defined by $f(x) = x + 1/x^2 + 1$ and $g(x) = x^3$. 
Then one can easily see that $f \in F(e)$ and $g \in \mathcal{H}(Y)$. Now

$$gf^{-1}(x) = gf(x^{1/3})$$

$$= g\left(x^{1/3} + \frac{1}{x^{2/3} + 1}\right)$$

$$= x + \frac{3x^{2/3}}{x^{2/3} + 1} + \frac{3x^{1/3}}{(x^{2/3} + 1)^2} + \frac{1}{(x^{2/3} + 1)^3}.$$  

Since

$$\lim_{x \to \infty} \frac{1}{(x^{2/3} + 1)^3} = 1,$$

we see that $gf^{-1} \notin F(e)$.

The remainder of our study is directed toward understanding $\mathcal{H}_f^+(\mathbb{R})$ as a topological space.

4. Properties of $\mathcal{H}_f^+(\mathbb{R})$ and $\Box\mathbb{R}^\omega$

In this section we examine some of the topological properties of $\mathcal{H}_f^+(\mathbb{R})$ and see that they are similar to the corresponding properties of the box product $\Box\mathbb{R}^\omega$ (see [15], [18], [19]). Let us start with the fact that these spaces are homogeneous, that is, each point can be mapped to each other point by a homeomorphism on the space.

**Proposition 4.1.** The spaces $\mathcal{H}_f^+(\mathbb{R})$ and $\Box\mathbb{R}^\omega$ are homogeneous.

**Proof:** The space $\mathcal{H}_f^+(\mathbb{R})$ is homogeneous because it is a topological group by Theorem 3.2. To show that $\Box\mathbb{R}^\omega$ is homogeneous, let $x, y \in \Box\mathbb{R}^\omega$. Then if $h : \Box\mathbb{R}^\omega \to \Box\mathbb{R}^\omega$ is defined by $h(z)_n = z_n - x_n + y_n$ for all $z \in \Box\mathbb{R}^\omega$ and $n \in \omega$, we see that $h$ is a homeomorphism that maps $x$ to $y$. \(\Box\)

Next let us consider the global properties of weight, density, and cellularity (see [9]). The weight of a topological space, $w(X)$, is the minimum cardinality of a base for $X$. The density of $X$, $d(X)$, is the minimum cardinality of a dense subset of $X$. The cellularity of $X$, $c(X)$, is the maximum cardinality of a pairwise disjoint family of nonempty open subsets of $X$. For all spaces $X$, we have

$$c(X) \leq d(X) \leq w(X).$$
These properties of □ℝω are well known, but we discuss them briefly to illustrate the similarity to the corresponding properties of ℋf(ℝ).

Let us define the equivalence relations ≈ and ∼ on □ℝω in a way similar to their definitions on ℋf(ℝ) (so we use the same notations). Let ≈ be defined on □ℝω by x ≈ y provided that there exists an m ∈ ω with x_n = y_n for all n > m. Also, let ∼ be defined on □ℝω by x ∼ y provided that for every ε > 0 there exists an m ∈ ω with |x_n - y_n| < ε for all n > m. For each x ∈ □ℝω, let E(x) and F(x) be the equivalence classes of ≈ and ∼, respectively, that contain x.

It can be shown, much as it was in propositions 3.3 and 3.8, that for each x ∈ □ℝω, E(x) and F(x) are closed subspaces of □ℝω such that F(x) is open but E(x) is not. In fact, □ℝω is equal to the topological sum of the distinct members of {F(x) : x ∈ □ℝω}.

Let c be the cardinality of the continuum ℝ. We see that there are at least c distinct members of {F(x) : x ∈ □ℝω} because if x, y ∈ □ℝω are such that x_n = a and y_n = b for all n ∈ ω where a ≠ b, then F(x) ≠ F(y). This means that c(□ℝω) ≥ c. But w(□ℝω) ≤ c since □ℝω has a base of cardinality c consisting of sets of the form \( \prod_{m \in \omega} U_n \) where each U_m is an open interval with rational endpoints. Therefore, we have the following fact about □ℝω.

**Proposition 4.2.** The box product □ℝω satisfies

\[
c(□ℝω) = d(□ℝω) = w(□ℝω) = c.
\]

Let us establish an analogous proposition for ℋf(ℝ). First observe that ℋf(ℝ) is a subspace of C_f(ℝ), so that w(ℋf(ℝ)) ≤ w(C_f(ℝ)). It is shown in [8] that for all spaces X,

\[
c(C_f(X)) = d(C_f(X)) = w(C_f(X)).
\]

Therefore, w(ℋf(ℝ)) ≤ d(C_f(ℝ)). We can now complete the proof of the following fact about ℋf(ℝ).

**Proposition 4.3.** The space ℋf(ℝ) satisfies

\[
c(ℋf(ℝ)) = d(ℋf(ℝ)) = w(ℋf(ℝ)) = c.
\]

**Proof:** Since w(ℋf(ℝ)) ≤ d(C_f(ℝ)), we need to see that d(C_f(ℝ)) ≤ c. But there is an injection from C(ℝ) into ℝω because two functions in C(ℝ) are equal if and only if they are equal at all rational
numbers. Then since the cardinality of \(\mathbb{R}\omega\) is \(c\), we know that the cardinality of \(C(\mathbb{R})\) is \(c\), and therefore, \(d(C_f(\mathbb{R})) \leq c\).

To show that \(c \leq c(H_f^+(\mathbb{R}))\), first recall from Corollary 3.9 that \(H_f^+(\mathbb{R})\) is equal to the topological sum of the distinct members of \(\{F(h) : h \in H_f^+(\mathbb{R})\}\). If \(f, g \in H_f^+(\mathbb{R})\) are such that \(f(t) = at\) and \(g(t) = bt\) for nonzero \(a \neq b\), then \(F(f) \neq F(g)\). This shows that there are at least \(c\) distinct members of \(\{F(h) : h \in H_f^+(\mathbb{R})\}\), and hence \(c \leq c(H_f^+(\mathbb{R}))\). □

We now turn to the local property of the character \(\chi(X)\) of a space \(X\), by which we mean the maximum, as \(x\) ranges over \(X\), of the minimum cardinality of a local base at \(x\). Since \(H_f^+(\mathbb{R})\) and \(\square\mathbb{R}\omega\) are homogeneous by Proposition 4.1, we need only to consider local bases at \(e\) in \(H_f^+(\mathbb{R})\) and at \(0\) in \(\square\mathbb{R}\omega\).

A subset \(D\) of \(\mathbb{R}\omega\) is said to be dominating provided that for each \(x\) in \(\mathbb{R}\omega\), there exists a \(d\) in \(D\) such that \(x_n \leq d_n\) for all \(n \in \omega\). The domination number, \(d\), is the minimum cardinality of a dominating subset of \(\mathbb{R}\omega\) (see [12]). This cardinal number \(d\) lies between the two cardinal numbers \(\aleph_1\) and \(2^{\aleph_0} = c\), and it is consistent with ZFC that it be equal to either one of these numbers or to neither of them (see [11]). By considering a local base of \(\square\mathbb{R}\omega\) at \(0\) and by taking reciprocals of the positive elements of a dominating subset of \(\mathbb{R}\omega\), we see the following fact.

**Proposition 4.4.** The box product \(\square\mathbb{R}\omega\) satisfies \(\chi(\square\mathbb{R}\omega) = d\).

Let us prove the corresponding property of \(H_f^+(\mathbb{R})\).

**Proposition 4.5.** The space \(H_f^+(\mathbb{R})\) satisfies \(\chi(H_f^+(\mathbb{R})) = d\).

**Proof:** From [8] we know that \(\chi(C_f(\mathbb{R})) = d\). Since \(H_f^+(\mathbb{R})\) is a subspace of \(C_f(\mathbb{R})\), it follows that \(\chi(H_f^+(\mathbb{R})) \leq d\).

We sketch an argument showing that \(d \leq \chi(H_f^+(\mathbb{R}))\). Let \(D_1(\mathbb{R})\) be defined as in Example 3.7. Then for every \(\varepsilon \in C_+(\mathbb{R})\), there exists a \(\delta \in C_+(\mathbb{R}) \cap D_1(\mathbb{R})\) such that \(\delta < \varepsilon\). This means the family of sets \(B(e, \delta)\) for all \(\delta \in D_1(\mathbb{R})\) forms a base at \(e\) in \(H_f^+(\mathbb{R})\). Also for each \(\delta \in D_1(\mathbb{R}), e + \delta \in H_f^+(\mathbb{R})\).

Now let \(\Delta \subseteq D_1(\mathbb{R})\) be such that \(\{B(e, \delta) : \delta \in \Delta\}\) is a base at \(e\) in \(H_f^+(\mathbb{R})\) with the cardinality of \(\Delta\) equal to \(\chi(H_f^+(\mathbb{R}))\). Define
For each Proposition 4.6, given in [6] as follows.

\[ \chi(D) \] is a dominating subset of \( C(\mathbb{R}) \).

To show that \( D \) is a dominating subset of \( C(\mathbb{R}) \), let \( f \in C(\mathbb{R}) \). Then there is an \( \varepsilon \in C_+ (\mathbb{R}) \) such that \( f \leq \varepsilon \). Since \( B(e, 1/\varepsilon) \) is a neighborhood of \( e \), there exists a \( \delta \in \Delta \) with \( B(e, \delta) \subseteq B(e, 1/\varepsilon) \).

To show that \( \delta \leq 1/\varepsilon \), suppose not. Then there exists an \( x \in \mathbb{R} \) with \( \delta(x) > 1/\varepsilon(x) \). Let \( k = 1/(\delta(x)\varepsilon(x)) \), which is strictly between 0 and 1. Now \( k\delta \in D_1(\mathbb{R}) \), so that \( e + k\delta \in H(\mathbb{R}) \). Also \( e + k\delta \in B(e, \delta) \). But \( k\delta(x) = 1/\varepsilon(x) \), so that \( e + k\delta \notin B(e, 1/\varepsilon) \).

With this contradiction, we have \( \delta \leq 1/\varepsilon \), and hence \( \varepsilon \leq 1/\delta \). So \( D \) is a dominating subset of \( C(\mathbb{R}) \), and thus \( d \leq |D| \leq |\Delta| = \chi(H_f^+(\mathbb{R})) \).

From propositions 4.4 and 4.5, we see that \( H_f^+(\mathbb{R}) \) and \( \square\mathbb{R}^\omega \) are not first countable, and hence not metrizable.

Now let us examine the connectedness properties of \( H_f^+(\mathbb{R}) \) and \( \square\mathbb{R}^\omega \). First the connected components of the box product \( \square\mathbb{R}^\omega \) are given in [6] as follows.

**Proposition 4.6.** For each \( x \in \square\mathbb{R}^\omega \), the connected component (path-component) of \( \square\mathbb{R}^\omega \) containing \( x \) is \( E(x) \).

We prove the analogous result for \( H_f^+(\mathbb{R}) \) in a way that can also be used to prove Proposition 4.6, but is different from that used in [6].

**Proposition 4.7.** For each \( h \in H_f^+(\mathbb{R}) \), the connected component (path-component) of \( H_f^+(\mathbb{R}) \) containing \( h \) is \( E(h) \).

**Proof:** We prove this for \( h = e \). Let \( f \in H_f^+(\mathbb{R}) \setminus E(e) \). Suppose, by way of contradiction, that \( f \) is in the connected component of \( H_f(\mathbb{R}) \) containing \( e \). Since \( f \neq E(e) \), there exists an increasing unbounded sequence \( \langle y_n \rangle \) in \( \mathbb{R} \) such that \( f(y_n) \neq y_n \) for all \( n \). For each \( n \), let \( \delta_n = |f(y_n) - y_n|/n \), and let \( \varepsilon \in C_+ (\mathbb{R}) \) be such that \( \varepsilon(y_n) = \delta_n \) for all \( n \). Then the open cover \( \{ B(g, \varepsilon) : g \in H_f(\mathbb{R}) \} \) of \( H_f(\mathbb{R}) \) has a simple chain connecting \( e \) to \( f \), say \( B(g_1, \varepsilon), \ldots, B(g_k, \varepsilon) \) where \( g_1 = e, g_k = f \), and \( B(g_i, \varepsilon) \cap B(g_j, \varepsilon) \neq \emptyset \) if and only if \( |i - j| \leq 1 \). Let \( n = 2k \), and for each \( i = 1, \ldots, k - 1 \), let

\[
\begin{align*}
z_i & \in B(g_i(y_n), \varepsilon(y_n)) \cap B(g_{i+1}(y_n), \varepsilon(y_n)) \\
& = B(g_i(y_n), \delta_n) \cap B(g_{i+1}(y_n), \delta_n).
\end{align*}
\]
Then we have
\[ 2k\delta_n = d(y_n, f(y_n)) \]
\[ \leq d(g_1(y_n), z_1) + d(z_1, g_2(y_n)) + d(g_2(y_n), z_2) + d(z_2, g_3(y_n)) \]
\[ + \cdots + d(g_{k-1}(y_n), z_{k-1}) + d(z_{k-1}, g_k(y_n)) \]
\[ < 2(k-1)\delta_n, \]
which is a contradiction. This shows that \( f \) is not in the connected component of \( \mathcal{H}_f(\mathbb{R}) \) containing \( e \), and hence this component is contained in \( E(e) \).

It remains to show that \( E(e) \) is connected. We need to show that for each \( f \in E(e) \), \( \{e, f\} \) is contained in some connected subset of \( E(e) \). So let \( f \in E(e) \). Define \( p : [0, 1] \to C(\mathbb{R}) \) by
\[ p(t)(y) = tf(y) + (1-t)y \]
for all \( t \in [0, 1] \) and \( y \in \mathbb{R} \). Clearly, \( p(0) = e \) and \( p(1) = f \). To show \( p(t) \in \mathcal{H}_f(\mathbb{R}) \) for each \( t \in [0, 1] \), we need only show that \( p(t) \) is increasing. But since \( f \) is increasing, it is evident that each \( p(t) \) is increasing. So \( p \) is a well-defined function from the interval \( [0, 1] \) into \( \mathcal{H}_f(\mathbb{R}) \).

Now \([0, 1] \) is connected in the usual topology, so we need to know that \( p \) is continuous (i.e., \( p \) is a path). Because \( f \in E(e) \), there exists a compact subset \( K \) of \( \mathbb{R} \) such that \( f(y) = y \) for all \( y \in \mathbb{R} \setminus K \). That means it suffices to think of \( p \) as a mapping from \([0, 1]\) into \( \mathcal{H}_f(K) \). But the fine topology on \( \mathcal{H}(K) \) is equal to the compact-open topology on \( \mathcal{H}(K) \), and \( p \) is continuous as a function into \( \mathcal{H}_k(K) \).

We point out that Proposition 4.7 is also true more generally for \( C_f(X) \) whenever \( X \) is a locally compact \( \sigma \)-compact space. Basically the same proof works in the more general setting.

The properties given above for \( \mathcal{H}_f^+(\mathbb{R}) \) and \( \square \mathbb{R}^\omega \) are so similar that one might wonder whether these spaces are homeomorphic. As it turns out, \( \mathcal{H}_f^+(\mathbb{R}) \) and \( \square \mathbb{R}^\omega \) differ in one important aspect, as we see from the following two propositions.

**Proposition 4.8.** The space \( \mathcal{H}_f^+(\mathbb{R}) \) contains a closed subspace that is homeomorphic to \( \mathbb{R}^\omega \).

**Proof:** Let \( H = \{ h \in \mathcal{H}^+(\mathbb{R}) : h(t) = t \text{ for all } t \in (-\infty, -1] \cup [1, \infty) \} \). One can easily check that \( H \) is closed in \( \mathcal{H}_p^+(\mathbb{R}) \), and hence
closed in $\mathcal{H}_f^+(\mathbb{R})$. Also it is evident that $H$, as a subspace of $\mathcal{H}_f^+(\mathbb{R})$, is homeomorphic to $\mathcal{H}_f^+(\mathbb{I}) = \mathcal{H}_k^+(\mathbb{I})$. But $\mathcal{H}_k^+(\mathbb{I})$ is homeomorphic to $\mathbb{R}^\omega$ by Theorem 2.2.

**Proposition 4.9.** The box product $\Box \mathbb{R}^\omega$ does not contain a closed subspace that is homeomorphic to $\mathbb{R}^\omega$.

**Proof:** Suppose there were to exist a closed embedding $\phi : \mathbb{R}^\omega \to \Box \mathbb{R}^\omega$. Since $\Box \mathbb{R}^\omega$ is homogeneous by Proposition 4.1, we may assume that 0 is in $\phi(\mathbb{R}^\omega)$. From Proposition 4.6 we know that $E(0)$ is the connected component of $\Box \mathbb{R}^\omega$ that contains 0, so that $\phi(\mathbb{R}^\omega) \subseteq E(0)$. But $E(0)$ is $\sigma$-compact and $\phi$ is a closed embedding, which contradicts the fact that $\mathbb{R}^\omega$ is not $\sigma$-compact.

**Corollary 4.10.** The space $\mathcal{H}_f^+(\mathbb{R})$ cannot be embedded as a closed subspace of the box product $\Box \mathbb{R}^\omega$.

So we see from Corollary 4.10 that $\mathcal{H}_f^+(\mathbb{R})$ is not homeomorphic to $\Box \mathbb{R}^\omega$. But $\mathcal{H}(\mathbb{R})$ has properties that are so close to those of $\Box \mathbb{R}^\omega$ that one might wonder whether $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\mathbb{R}^\omega$ with some product topology that is slightly weaker than the box product topology, but stronger than the Tychonoff product topology. In the next section we define such a topology that seems to be the natural one.

### 5. Semi-box product spaces

For a topological space $X$, we define the *semi-box product space* $\Box X^\omega$ to be the product $X^\omega$ with the *semi-box product topology* that we define as follows. Let $Y$ be a separable metric space that is dense in itself (i.e., it has no isolated point), and let $A$ be a nonempty proper compact subset of $Y$. Let $\phi$ be a bijection from the set of finite ordinals $\omega$ onto a dense subset of $Y$. Let $S_1$ ($S_2$, respectively) be the set of subsets $S$ of $\omega$ such that the set of accumulation points of $\phi(S)$ in $Y$ is contained in $A$ (is equal to $A$, respectively); and let $i \in \{1, 2\}$. Since $S_i$ is a cover of $\omega$ that is closed under finite unions, the following is a base for a topology on $X^\omega$. The semi-box product topology on $X^\omega$ has a base consisting of sets of the form

$$\prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} X_m,$$
where $S \in S_i$, each $X_m$ is a copy of $X$, and each $U_m$ is an open subset of $X_m$. This definition also applies if the $X_m$ are different for different $m$, in which case we denote this semi-box product space by $\sqcup_{n \in \omega} X_m$.

**Theorem 5.1.** For a topological space $X$, the semi-box product topology on $X^\omega$ is independent of the choice of $Y$, $A$, $\phi$, and $i$ in the definition above.

The proof of Theorem 5.1 follows from the next three lemmas.

**Lemma 5.2.** The semi-box product topology is independent of the choice of $i$.

*Proof:* First, since $S_2 \subseteq S_1$, the semi-box product topology on $X^\omega$ using $S_1$ is finer than or equal to the semi-box product topology on $X^\omega$ using $S_2$.

Next we show that $S_1$ actually refines $S_2$; so let $S_1 \in S_1$. We need to find an $S_2 \in S_2$ with $S_1 \subseteq S_2$. For each $m \in \omega \setminus \{0\}$, let $\mathcal{V}_m$ be a finite open cover of $A$ in $Y$ consisting of sets of diameter less than $1/m$. We define by induction a family $\{T_m : m \in \omega\}$ of finite subsets of $\omega$. First, let $T_0 = \emptyset$. Suppose $T_0, \ldots, T_m$ have been defined. Then for each $V \in \mathcal{V}_{m+1}$, let

$$V' = V \setminus \left( \phi(S_1) \cup \phi(T_0) \cup \cdots \cup \phi(T_m) \right),$$

and let $\mathcal{V}'_{m+1} = \{V \in \mathcal{V}_{m+1} : V' \neq \emptyset\}$. For each $V \in \mathcal{V}'_{m+1}$, let $n_V \in \omega$ be such that $\phi(n_V) \in V'$. Then take $T_{m+1} = \{n_V : V \in \mathcal{V}'_{m+1}\}$.

With the family $\{T_m : m \in \omega\}$ thus defined by induction, now define

$$S_2 = S_1 \cup \cup\{T_m : m \in \omega\}.$$

We need to show that $S_2 \in S_2$. Now the set of accumulation points of $\phi(S_1)$ is contained in $A$. If $S$ is a sequence contained in $S_2 \setminus S_1$, then by the construction above, $\phi(S)$ is a Cauchy sequence and must converge to a point of $A$. So the set of accumulation points of $\phi(S_2)$ is contained in $A$. Finally, if $a \in A$, we again see from the construction above that there is a sequence $S$ in $S_2$ such that $\phi(S)$ converges to $a$. Therefore, the set of accumulation points of $\phi(S_2)$ is equal to $A$, and hence $S_2 \in S_2$, which completes the proof that $S_1$ refines $S_2$. 

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It now follows that for a typical basic open set
\[ \prod_{m \in S_1} U_m \times \prod_{m \in \omega \setminus S_1} X_m \]
for $\sqcap X^\omega$ using $S_1$, we can write this as
\[ \prod_{m \in S_2} U_m \times \prod_{m \in \omega \setminus S_2} X_m, \]
where $S_2$ is an element of $S_2$ containing $S_1$, and for each $m \in S_2 \setminus S_1$, $U_m = X_m$. This shows that the semi-box product topology on $X^\omega$ using $S_2$ is finer than or equal to the semi-box product topology on $X^\omega$ using $S_1$, and thus these topologies are equal. \(\square\)

We now use the notation $S$ to stand for either $S_1$ or $S_2$, since Lemma 5.2 tells us that it does not matter which is used.

**Lemma 5.3.** The semi-box product topology is independent of the choice of $\phi$.

**Proof:** Let $\phi$ and $\phi'$ be bijections from $\omega$ onto dense subsets of $Y$. Let $\sqcap X^\omega$ be the semi-box product space using $\phi$ and let $\sqcap' X^\omega$ be the semi-box product space using $\phi'$. Let $d$ be the metric on $Y$.

We define, by induction, sequences $\langle m_{2n-1} \rangle$, $\langle m_{2n} \rangle$, $\langle m'_{2n-1} \rangle$, and $\langle m'_{2n} \rangle$ in $\omega$. First, let $m_1 = 0$, and let $m'_1$ be the smallest $m \in \omega$ such that $d(\phi(m_1), \phi'(m_1)) < 1$. Next, let $m'_2$ be the smallest element of $\omega \setminus \{m'_1\}$, and let $m_2$ be the smallest $m \in \omega \setminus \{m_1\}$ such that $d(\phi(m), \phi'(m'_2)) < 1/2$. Now suppose that $n$ is in the set $\mathbb{N}$ of positive integers with $n > 1$ and that $m_{2k-1}$, $m_{2k}$, $m'_{2k-1}$, and $m'_{2k}$ have been defined for $k = 1, \ldots, n - 1$. Then let $m_{2n-1}$ be the smallest element of $\omega \setminus \{m_1, \ldots, m_{2n-2}\}$, and let $m'_{2n-1}$ be the smallest $m \in \omega \setminus \{m'_1, \ldots, m'_{2n-2}\}$ such that $d(\phi(m_{2n-1}), \phi'(m)) < 1/(2n - 1)$. Also let $m'_{2n}$ be the smallest element of $\omega \setminus \{m'_1, \ldots, m'_{2n-1}\}$, and let $m_{2n}$ be the smallest $m \in \omega \setminus \{m_1, \ldots, m_{2n-1}\}$ such that $d(\phi(m), \phi'(m'_2)) < 1/(2n)$. This completes the inductive definition of these sequences.

Note that for all $i, j \in \mathbb{N}$ with $i \neq j$, we have $m_i \neq m_j$ and $m'_i \neq m'_j$, and that $d(\phi(m_i), \phi'(m'_j)) < 1/i$. Also $\{m_n : n \in \mathbb{N}\} = \{m'_n : n \in \mathbb{N}\} = \omega$. Now define bijection $\psi$ from $\omega$ onto itself by $\psi(m_n) = m'_n$ for all $n \in \mathbb{N}$.

The construction above ensures, for each subset $S$ of $\omega$, that $\phi(S)$ and $\phi'(\psi(S))$ have the same set of accumulation points in $Y$, and
that \( \phi'(S) \) and \( \phi(\psi^{-1}(S)) \) have the same set of accumulation points in \( Y \). In particular, if \( S \) is taken using \( \phi \) and \( S' \) is taken using \( \phi' \), we have \( \{ \psi(S) : S \in S \} = S' \) and \( \{ \psi^{-1}(S) : S \in S' \} = S \).

Now define \( \Psi : \boxdot X^\omega \to \boxdot' X^\omega \) by \( \Psi(x)_m = x_{\psi(m)} \) for all \( x \) in \( \boxdot X^\omega \) and all \( m \in \omega \). Then \( \Psi \) is a bijection because \( \psi \) is a bijection. Also, for \( S \in S \),

\[
\Psi \left( \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} X_m \right) = \prod_{m \in \psi(S)} U_m \times \prod_{m \in \omega \setminus \psi(S)} X_m;
\]

and for \( S \in S' \),

\[
\Psi^{-1} \left( \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} X_m \right) = \prod_{m \in \psi^{-1}(S)} U_m \times \prod_{m \in \omega \setminus \psi^{-1}(S)} X_m.
\]

This shows that \( \Psi \) is a homeomorphism from \( \boxdot X^\omega \) onto \( \boxdot' X^\omega \). □

**Lemma 5.4.** The semi-box product topology is independent of the choice of \( Y \) and \( A \).

**Proof:** Let \( Y' \) and \( A' \) be another admissible pair; let \( \boxdot X^\omega \) be the semi-box product space using \( Y', A, \phi, \) and \( S \); and let \( \boxdot' X^\omega \) be the semi-box product space using \( Y', A', \phi', \) and \( S' \).

We can choose sequences \( \{ V_n \} \) and \( \{ V'_n \} \) of finite open covers of \( A \) in \( Y \) and \( A' \) in \( Y' \), respectively, having the following properties. For each \( n \in \mathbb{N} \),

\[
V_n = \{ V_{n,1}, \ldots, V_{n,k_n} \}
\]

and

\[
V'_n = \{ V'_{n,1}, \ldots, V'_{n,k_n} \},
\]

where for each \( j = 1, \ldots, k_n \), \( V_{n,j} \) and \( V'_{n,j} \) are open balls of radius \( \varepsilon_n \) centered at points in \( A \) and \( A' \), respectively (the centers do not have to be distinct). Also, the sequence \( \{ \varepsilon_n \} \) is a decreasing sequence converging to 0, \( \overline{\bigcup V_1} \) and \( \overline{\bigcup V'_1} \) are proper subsets of \( Y \) and \( Y' \), and for each \( n \), \( \overline{\bigcup V_{n+1}} \) and \( \overline{\bigcup V'_{n+1}} \) are proper subsets of \( \overline{\bigcup V_n} \) and \( \overline{\bigcup V'_n} \), respectively.

Let \( S_0 = \phi^{-1}(Y \setminus \bigcup V_1) \) and \( S'_0 = \phi'^{-1}(Y \setminus \bigcup V'_1) \). We define, by induction, sequences \( \langle S_{2n-1} \rangle, \langle S'_{2n-1} \rangle, \langle S_{2n} \rangle, \text{ and } \langle S'_{2n} \rangle \) of subsets of \( \omega \) as follows. Suppose for \( n \in \mathbb{N} \), \( S_0, S_2, \ldots, S_{2n-2} \), and \( S'_0, S'_2, \ldots, S'_{2n-2} \) have been defined. To define \( S_{2n-1} \), let \( m_1 \) be the first element of \( \phi^{-1}(V_{n,1}) \setminus (S_0 \cup \cdots \cup S_{2n-2}) \), let \( m_2 \) be the first element of \( \phi'^{-1}(V_{n,2}) \setminus (S'_0 \cup \cdots \cup S'_{2n-2} \cup \{ m_1 \}) \), and so on up to \( m_{k_n} \). Then
let $S_{2n-1} = \{m_1, \ldots, m_{k_n}\}$. In a similar way, define $m'_1, \ldots, m'_{k_n}$ in $\omega$ using $S'_0, \ldots, S'_{2n-2}$, $\phi'$, and $V'_{n,1}, \ldots, V'_{n,k_n}$; and let $S'_{2n-1} = \{m'_1, \ldots, m'_{k_n}\}$. Also let $S_{2n} = \phi^{-1}(\cup V_n \cup V_{n+1}) \setminus (S_0 \cup \cdots \cup S_{2n-1})$ and $S'_{2n} = \phi'^{-1}(\cup V'_n \cup V'_{n+1}) \setminus (S'_0 \cup \cdots \cup S'_{2n-1})$.

With $S_m$ and $S'_m$ thus defined for all $m \in \omega$, we see that $\{S_m : m \in \omega\}$ and $\{S'_m : m \in \omega\}$ each forms a partition of $\omega$. Also for every $n \in \mathbb{N}$, $S_{2n-1}$ and $S'_{2n-1}$ each has $k_n$ elements, while each $S_{2n-2}$ and $S'_{2n-2}$ is infinite. So we can define a bijection $\psi$ from $\omega$ onto itself by mapping each $S_{2n-1}$ onto $S'_{2n-1}$ and each $S_{2n-2}$ onto $S'_{2n-2}$.

For the construction above, we see that $\{\psi(S) : S \in \mathcal{S}\} = \mathcal{S}'$ and $\{\psi^{-1}(S) : S \in \mathcal{S}'\} = \mathcal{S}$. So using the argument in the last paragraph of the proof of Lemma 5.3, we get a homeomorphism $\Psi$ from $\Box X^\omega$ onto $\Box' X^\omega$. \qed

For all spaces $X$, the semi-box product topology on $X^\omega$ is finer than or equal to the Tychonoff product topology on $X^\omega$ and is coarser than or equal to the box product topology on $X^\omega$. In the next section we examine the properties of $\Box \mathbb{R}^\omega$, and in particular, we see that the topology on $\Box \mathbb{R}^\omega$ is strictly finer than that on $\mathbb{R}^\omega$ and strictly coarser than that on $\Box \mathbb{R}^\omega$.

6. Properties of $\Box \mathbb{R}^\omega$

The topological properties of $\Box \mathbb{R}^\omega$ are similar to those properties of $\Box \mathbb{R}^\omega$ and $\mathcal{H}_f^+(\mathbb{R})$ that were studied in section 4. Throughout this section, $\mathcal{S}$ is used with $\Box \mathbb{R}^\omega$ as given by the definition of the semi-box product topology.

We define the equivalence relations $\approx$ and $\sim$ on $\Box \mathbb{R}^\omega$ in a way that is similar to their definitions on $\Box \mathbb{R}^\omega$, except we need to take into account the members of $\mathcal{S}$. For each $S \in \mathcal{S}$, let $\approx_S$ be defined on $\Box \mathbb{R}^\omega$ by $x \approx_S y$, provided that there exists an $m \in S$ such that $x_n = y_n$ for all $n \in S$ with $n > m$. Also for each $S \in \mathcal{S}$, let $\sim_S$ be defined on $\Box \mathbb{R}^\omega$ by $x \sim_S y$, provided that for each $\varepsilon > 0$ there exists an $m \in S$ such that $|x_n - y_n| < \varepsilon$ for all $n \in S$ with $n > m$. For each $S \in \mathcal{S}$ and each $x \in \Box \mathbb{R}^\omega$, let $E_S(x)$ and $F_S(x)$ be the equivalence classes of $\approx_S$ and $\sim_S$, respectively, that contain $x$. Now define $x \approx_S y$, provided that $x \approx_S y$ for all $S \in \mathcal{S}$. Then for each
The equivalence class of \( \approx \) that contains \( x \) is given by \( E(x) = \cap \{E_S(x) : S \in \mathcal{S}\} \). One can define \( \sim \) and \( F(x) \) in a similar manner.

As was true in \( \boxtimes \mathbb{R}^\omega \), it is also true in \( \boxplus \mathbb{R}^\omega \) that for each \( S \in \mathcal{S} \) and each \( x \in \boxplus \mathbb{R}^\omega \), \( E_S(x) \) and \( F_S(x) \) are closed in \( \boxplus \mathbb{R}^\omega \), and that \( F_S(x) \) is also open but \( E_S(x) \) is not. It follows that \( E(x) \) and \( F(x) \) are closed in \( \boxplus \mathbb{R}^\omega \); however, it does not follow that \( F(x) \) is open in \( \boxplus \mathbb{R}^\omega \).

The properties of \( \boxplus \mathbb{R}^\omega \) are summarized by the next proposition, and we see that these are essentially the same properties that \( \boxtimes \mathbb{R}^\omega \) and \( \mathcal{H}^+_{\mathbb{R}} \) have.

**Proposition 6.1.** The semi-box product \( \boxplus \mathbb{R}^\omega \) satisfies the following.

- (1) \( \boxplus \mathbb{R}^\omega \) is homogeneous.
- (2) For every \( S \in \mathcal{S} \), \( \boxplus \mathbb{R}^\omega \) is equal to the topological sum of the distinct members of \( \{F_S(x) : x \in \boxplus \mathbb{R}^\omega\} \).
- (3) For every \( x \in \boxplus \mathbb{R}^\omega \), the connected component (path-component) of \( \boxplus \mathbb{R}^\omega \) containing \( x \) is \( E(x) \).
- (4) \( c(\boxplus \mathbb{R}^\omega) = d(\boxplus \mathbb{R}^\omega) = w(\boxplus \mathbb{R}^\omega) = c \).
- (5) \( \chi(\boxplus \mathbb{R}^\omega) = d \).

**Proof:** The proof of (1) is the same as that used for \( \boxtimes \mathbb{R}^\omega \) in Proposition 4.1. Also the proof of (2) is evident since each \( F_S(x) \) is open and closed in \( \boxplus \mathbb{R}^\omega \).

For the proof of (3), an argument like that in Proposition 4.7 shows that if \( y \in \boxplus \mathbb{R}^\omega \setminus E(x) \), then \( y \) cannot be in the connected component of \( \boxplus \mathbb{R}^\omega \) containing \( x \). It remains to show that \( E(x) \) is connected. By Theorem 5.1, we can assume that in the definition of \( \boxplus \mathbb{R}^\omega \), \( Y = \mathbb{I} \) and \( A = \{-1, 1\} \). Then for each \( i \in \mathbb{N} \), let

\[
C_i = \{y \in \boxplus \mathbb{R}^\omega : y_n = x_n \text{ for all } n \in \omega \setminus \phi^{-1}([-1 + 1/i, 1 - 1/i])\}.
\]

Now each \( C_i \) is homeomorphic to \( \mathbb{R}^\omega \) and is thus connected. Then since each \( C_i \) contains \( x \), \( \cup\{C_i : i \in \mathbb{N}\} \) must be connected. But \( \cup\{C_i : i \in \mathbb{N}\} = E(x) \), so that \( E(x) \) is connected.

For the proof of (4), we can use (2) to see that \( c(\boxtimes \mathbb{R}^\omega) \geq c \). But also the topology on \( \boxplus \mathbb{R}^\omega \) is coarser than or equal to the topology on \( \boxtimes \mathbb{R}^\omega \). So since \( w(\boxtimes \mathbb{R}^\omega) = c \) by Proposition 4.2, we have \( w(\boxplus \mathbb{R}^\omega) \leq c \). This shows that \( c(\boxplus \mathbb{R}^\omega) = d(\boxplus \mathbb{R}^\omega) = w(\boxplus \mathbb{R}^\omega) = c \).
Finally, for the proof of (5), since \(\chi(\square \mathbb{R}^\omega) = d\) from Proposition 4.4, we have \(\chi(\square \mathbb{R}^\omega) \leq d\). But also, for any \(S \in \mathcal{S}\), the closed subspace
\[
\{x \in \square \mathbb{R}^\omega : x_m = 0 \text{ for all } m \in \omega \setminus S\}
\]
of \(\square \mathbb{R}^\omega\) is homeomorphic to \(\square \mathbb{R}^\omega\). This shows that \(\chi(\square \mathbb{R}^\omega) \geq d\), and hence \(\chi(\square \mathbb{R}^\omega) = d\). \(\square\)

In the proof above, notice that closed subspaces \(C_i\) of \(\square \mathbb{R}^\omega\) are given that are homeomorphic to \(\mathbb{R}^\omega\). As shown by Proposition 4.9, this is the big difference between \(\square \mathbb{R}^\omega\) and \(\square \mathbb{R}^\omega\) which does not have closed subspaces homeomorphic to \(\mathbb{R}^\omega\). This fact, along with properties (2) through (5) in Proposition 6.1, shows the following.

**Proposition 6.2.** The topology on \(\square \mathbb{R}^\omega\) is strictly finer than that on \(\mathbb{R}^\omega\) and is strictly coarser than that on \(\square \mathbb{R}^\omega\).

The semi-box product \(\square \mathbb{R}^\omega\) also has both \(\mathbb{R}^\omega\) and \(\square \mathbb{R}^\omega\) as factors, as shown in the next proposition.

**Proposition 6.3.** The semi-box product \(\square \mathbb{R}^\omega\) is homeomorphic to both \(\mathbb{R}^\omega \times \square \mathbb{R}^\omega\) and \(\square \mathbb{R}^\omega \times \square \mathbb{R}^\omega\).

**Proof:** Let us first map \(\square \mathbb{R}^\omega\) onto \(\mathbb{R}^\omega \times \square \mathbb{R}^\omega\). For the \(\square \mathbb{R}^\omega\) in the domain of our map, we use \(Y = \mathbb{I}, A = \{-1, 1\}\), and \(\phi\) any bijection from \(\omega\) onto a dense subset of \(\mathbb{I}\). Let \(J = (-1/2, 1/2)\), let \(\phi_1\) be a bijection from \(\omega\) onto \(\phi(\omega) \cap J\), and let \(\phi_2\) be a bijection from \(\omega\) onto \(\phi(\omega) \setminus J\). For the \(\square \mathbb{R}^\omega\) in the range of our map, we use \(Y = \mathbb{I} \setminus J, A = \{-1, 1\}\), and \(\phi_2\).

Define \(\Gamma : \square \mathbb{R}^\omega \to \mathbb{R}^\omega \times \square \mathbb{R}^\omega\) as follows. For each \(x \in \square \mathbb{R}^\omega\), let \(\Gamma(x) = (y, z) \in \mathbb{R}^\omega \times \square \mathbb{R}^\omega\) where \(y\) and \(z\) are such that
\[
y_m = x_{\phi_1^{-1}(\phi_1(m))}\text{ for all } m \in \omega
\]
and
\[
z_m = x_{\phi_2^{-1}(\phi_2(m))}\text{ for all } m \in \omega.
\]
We see that \(\Gamma\) is a bijection having inverse \(\Gamma^{-1} : \mathbb{R}^\omega \times \square \mathbb{R}^\omega \to \square \mathbb{R}^\omega\) defined as follows. For each \((y, z) \in \mathbb{R}^\omega \times \square \mathbb{R}^\omega\), let \(\Gamma^{-1}((y, z)) = x \in \square \mathbb{R}^\omega\) where \(x\) is such that
\[
x_m = y_{\phi_1^{-1}(\phi(m))}\text{ for all } m \in \phi^{-1}(\phi(\omega) \cap J)
\]
and
\[
x_m = z_{\phi_2^{-1}(\phi(m))}\text{ for all } m \in \phi^{-1}(\phi(\omega) \setminus J).
\]
To show that $\Gamma$ is continuous, let $x \in \sqcap \mathbb{R}^\omega$ and let $\langle y, z \rangle = \Gamma(x)$. Also let $B(y, F, \varepsilon)$ be a basic neighborhood of $y$ in $\mathbb{R}^\omega$ where $F$ is a finite subset of $\omega$ and $\varepsilon > 0$, and let

$$V = \prod_{m \in T} V_m \times \prod_{m \in \omega \setminus T} \mathbb{R}_m$$

be a basic neighborhood of $z$ in $\sqcap \mathbb{R}^\omega$ where $T \subseteq \omega$ is such that the set of accumulation points of $\phi_2(T)$ in $\mathbb{I} \setminus J$ is $A$. Define $S = \phi^{-1}_{\phi_1}(F) \cup \phi^{-1}_{\phi_2}(T)$, which has the property that the set of accumulation points of $\phi(S)$ in $\mathbb{I}$ is $A$. Also for each $m \in S$, let

$$U_m = \begin{cases} (y_{\phi_1^{-1}\phi(m)} - \varepsilon, y_{\phi_1^{-1}\phi(m)} + \varepsilon), & \text{if } m \in \phi^{-1}_{\phi_1}(F), \\ V_{\phi_2^{-1}\phi(m)}, & \text{if } m \in \phi^{-1}_{\phi_2}(T). \end{cases}$$

Now define

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m,$$

which is a neighborhood of $x$ in $\sqcap \mathbb{R}^\omega$. We see that $\Gamma(U) \subseteq B(y, F, \varepsilon) \times V$, showing that $\Gamma$ is continuous.

To show that $\Gamma^{-1}$ is continuous, let $\langle y, z \rangle \in \mathbb{R}^\omega \times \sqcap \mathbb{R}^\omega$, let $x = \Gamma^{-1}(\langle y, z \rangle)$, and let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m$$

be a basic neighborhood of $x$ in $\sqcap \mathbb{R}^\omega$. Define $F = \phi^{-1}_{\phi_1}(\phi(S) \cap J)$ and $T = \phi^{-1}_{\phi_2}(\phi(S) \setminus J)$. Since the set of accumulation points of $\phi(S)$ in $\mathbb{I}$ is $A$, $F$ must be finite and the set of accumulation points of $\phi_2(T)$ in $\mathbb{I} \setminus J$ is $A$. Now define $\varepsilon = \min\{\varepsilon_{\phi^{-1}_{\phi_1}(m)} : m \in F\}$, so that $B(y, F, \varepsilon)$ is a neighborhood of $y$ in $\mathbb{R}^\omega$. Also define

$$V = \prod_{m \in T} V_m \times \prod_{m \in \omega \setminus T} \mathbb{R}_m$$

where $V_m = U_{\phi^{-1}_{\phi_2}(m)}$ for all $m \in T$, which is a neighborhood of $z$ in $\sqcap \mathbb{R}^\omega$. We see that $\Gamma^{-1}(B(y, F, \varepsilon) \times V) \subseteq U$, showing that $\Gamma^{-1}$ is continuous.

For the map of $\sqcap \mathbb{R}^\omega$ onto $\sqcap \mathbb{R}^\omega \times \sqcap \mathbb{R}^\omega$, we use the same map $\Gamma$, but we let $J$ be any subset of $\phi(\omega)$ whose set of accumulation points in $\mathbb{I}$ is $A$, let $\phi_1$ be a bijection from $\omega$ onto $J$, and let $\phi_2$ be a bijection from $\omega$ onto $\phi(\omega) \setminus J$. Now using exactly the same
definition for \( \Gamma \), we can argue in a similar way to show that \( \Gamma \) is a homeomorphism from \( \square \mathbb{R}^\omega \) onto \( \square \mathbb{R}^\omega \times \square \mathbb{R}^\omega \). \( \square \)

Now with the goal of discovering how \( \mathcal{H}_f^+ (\mathbb{R}) \) is related to \( \square \mathbb{R}^\omega \), we establish the following fact, which also pertains to the more general space \( C_f^+ (\mathbb{R}) \).

**Lemma 6.4.** Let \( D \) be a dense subset of \( \mathbb{R} \). Then the space \( C_f^+ (\mathbb{R}) \) has a base consisting of the sets of the form

\[
B(f, T, \varepsilon) = \{ g \in C_f^+ (\mathbb{R}) : |f(t) - g(t)| < \varepsilon(t) \text{ for all } t \in T \}
\]

where \( f \in C_f^+ (\mathbb{R}) \), \( T \) is a countable closed discrete subset of \( \mathbb{R} \) contained in \( D \), and \( \varepsilon \in C_+ (\mathbb{R}) \).

**Proof:** To show that \( B(f, T, \varepsilon) \) is open in \( C_f^+ (\mathbb{R}) \), let \( g \in B(f, T, \varepsilon) \). For each \( t \in T \), let \( \delta(t) = \varepsilon(t) - |g(t) - f(t)| \). Since \( T \) is closed and discrete in \( \mathbb{R} \), there exists a \( \delta \in C_+ (\mathbb{R}) \) whose value at each \( t \) is \( \delta(t) \). Let \( h \in B(g, \delta) \). Then for each \( t \in T \),

\[
|h(t) - f(t)| \leq |h(t) - g(t)| + |g(t) - f(t)| \\
< \delta(t) + (\varepsilon(t) - \delta(t)) \\
= \varepsilon(t).
\]

Therefore, \( h \in B(f, T, \varepsilon) \), so that \( B(g, \delta) \subseteq B(f, T, \varepsilon) \). This shows that \( B(f, T, \varepsilon) \) is open in \( C_f^+ (\mathbb{R}) \).

Now let \( f \in C_f^+ (\mathbb{R}) \) and let \( \varepsilon > 0 \). We need to find a \( T \) and \( \delta \in C_+ (\mathbb{R}) \) such that \( B(f, T, \delta) \subseteq B(f, \varepsilon) \). Let \( n \in \mathbb{N} \), and consider \( C_f^+ ([n-1, n]) \), which is equal to \( C_k^+ ([n-1, n]) \). By Lemma 2.3, there exist a finite subset \( F_n \) of \([n-1, n] \cap D \) and a \( \delta_n \in C_+ ([n-1, n]) \) such that

\[
B(f|_{[n-1, n]}, F_n, \delta_n) \subseteq B(f|_{[n-1, n]}, \varepsilon|_{[n-1, n]}).
\]

Similarly, there exist a finite subset \( F_{n-} \) of \([-n, n+1] \cap D \) and a \( \delta_{n-} \in C_+ ([n-1, n+1]) \) such that

\[
B(f|_{[-n, n+1]}, F_{n-}, \delta_{n-}) \subseteq B(f|_{[-n, n+1]}, \varepsilon|_{[-n, n+1]}).
\]

Let \( D = \bigcup \{ F_n : n \in \mathbb{N} \} \cup \bigcup \{ F_{n-} : n \in \mathbb{N} \} \). Then \( D \) is closed and discrete in \( \mathbb{R} \) and contained in \( D \). Also there exists a \( \delta \in C_+ (\mathbb{R}) \) such that \( \delta(t) \leq \delta_n (t) \) for all \( t \in [n-1, n] \) and \( \delta(t) \leq \delta_{n-} (t) \) for all \( t \in [-n, n+1] \). Then \( B(f, T, \delta) \subseteq B(f, \varepsilon) \), showing that the sets of the form \( B(f, T, \delta) \) do form a base for \( C_f^+ (\mathbb{R}) \). \( \square \)
We can now relate $\mathcal{H}_j^+(\mathbb{R})$ to $\boxdot \mathbb{R}^\omega$ as follows.

**Theorem 6.5.** The space $\mathcal{H}_j^+(\mathbb{R})$ can be embedded into the semi-box product $\boxdot \mathbb{R}^\omega$, and in turn, the semi-box product $\boxdot \mathbb{R}^\omega$ can be embedded as a closed subspace of $\mathcal{H}_j^+(\mathbb{R})$. As a consequence of the latter embedding, the box product $\square \mathbb{R}^\omega$ can be embedded as a closed subspace of $\mathcal{H}_j^+(\mathbb{R})$.

**Proof:** In the definition of the semi-box product $\boxdot \mathbb{R}^\omega$, because of Theorem 5.1, we can take $Y = \mathbb{I}$ and $A = \{-1, 1\}$. We then have $\phi$ mapping $\omega$ onto a dense subset of $Y \setminus A$, and we have $S$ defined as in the definition.

To show that $\mathcal{H}_j^+(\mathbb{R})$ can be embedded into $\boxdot \mathbb{R}^\omega$, we define $\Lambda : \mathcal{H}_j^+(\mathbb{R}) \to \boxdot \mathbb{R}^\omega$ by

$$
\Lambda(h)_m = h\left(\tan(\pi \phi(m)/2)\right)
$$

for all $h \in \mathcal{H}_j^+(\mathbb{R})$ and $m \in \omega$. Now $\Lambda$ is one-to-one since if $h_1, h_2 \in \mathcal{H}_j^+(\mathbb{R})$ with $h_1 \neq h_2$, because the set

$$
D = \{\tan(\pi \phi(m)/2) : m \in \omega\}
$$

is dense in $\mathbb{R}$, there exists an $m \in \omega$ such that

$$
h_1\left(\tan(\pi \phi(m)/2)\right) \neq h_2\left(\tan(\pi \phi(m)/2)\right),
$$

showing that $\Lambda(h_1) \neq \Lambda(h_2)$.

To show that $\Lambda$ is continuous, let $h \in \mathcal{H}_j^+(\mathbb{R})$ and let

$$
U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m
$$

be a basic neighborhood of $\Lambda(h)$ in $\boxdot \mathbb{R}^\omega$. We can assume that for each $m \in S$, $U_m = (\Lambda(h)_m - \varepsilon_m, \Lambda(h)_m + \varepsilon_m)$ for some $\varepsilon_m > 0$. Since $\phi(S)$ is closed and discrete in $(-1, 1)$, the set $\{\tan(\pi \phi(m)/2) : m \in S\}$ is closed and discrete in $\mathbb{R}$. So there exists an $\varepsilon \in C_+(\mathbb{R})$ such that $\varepsilon(\tan(\pi \phi(m)/2)) = \varepsilon_m$ for all $m \in S$. Then

$$
\Lambda(B(h, \varepsilon)) \subseteq U,
$$

showing that $\Lambda$ is continuous.

Let $P = \Lambda(\mathcal{H}_j^+(\mathbb{R}))$. To show that $\Lambda^{-1} : P \to \mathcal{H}_j^+(\mathbb{R})$ is continuous, let $x \in P$. By Lemma 6.4, we can take as an arbitrary neighborhood of $\Lambda^{-1}(x)$ one that looks like $B(\Lambda^{-1}(x), T, \varepsilon)$ where $T$ is a countable closed discrete subset of $\mathbb{R}$ contained in $D$ and
ε ∈ C_+(ℝ). Now the set $D' = \{2\tan^{-1}(t)/\pi : t ∈ T\}$ is a countable closed discrete subset of $(-1, 1)$ contained in $ϕ(ω)$. So the set $S = ϕ^{-1}(D')$ is in $S$. Let

$$U = \prod_{m ∈ S} U_m \times \prod_{m ∈ ω \setminus S} \mathbb{R}_m$$

where each

$$U_m = (x_m - ε(\tan(πϕ(m)/2)), x_m + ε(\tan(πϕ(m)/2)))$$

Then

$$Λ(U \cap P) ⊆ B(Λ^{-1}(x), T, ε),$$

showing that $Λ^{-1}$ is continuous on $P$, and thus finishing the argument that $Λ$ is an embedding.

To obtain a closed embedding of $⊆ ℝ^ω$ into $ℋ^+_f(ℝ)$, we first assume that the image of $ϕ$ does not contain any $1 - 1/i$ or $-1 + 1/i$ for $i ∈ \mathbb{N}$. Then for each $i ∈ \mathbb{N}$, define

$$T_i = ϕ^{-1}([1 - 1/i, 1 - 1/(i + 1)] \cap ϕ(ω))$$

and

$$T_{-i} = ϕ^{-1}([-1 + 1/(i + 1), -1 + 1/i] \cap ϕ(ω)).$$

Note that $\{T_i : i ∈ \mathbb{N}\} \cup \{T_{-i} : i ∈ \mathbb{N}\}$ forms a partition of $ω$. For each $x ∈ ⊆ ℝ^ω$ and each $T ⊆ ω$, let $x_T$ denote the projection of $x$ into $\prod_{m ∈ T} \mathbb{R}_m$. Now for each $i ∈ \mathbb{N}$, $\prod_{m ∈ T_i} \mathbb{R}_m$ and $\prod_{m ∈ T_{-i}} \mathbb{R}_m$ have the Tychonoff product topology. So by Theorem 2.2, for each $i ∈ \mathbb{N}$, there exist homeomorphisms

$$α_i : \prod_{m ∈ T_i} \mathbb{R}_m → ℋ^+_k([i - 1, i])$$

and

$$α_{-i} : \prod_{m ∈ T_{-i}} \mathbb{R}_m → ℋ^+_k([-1, -1 + 1]).$$

Define $Φ : ⊆ ℝ^ω → ℋ^+_f(ℝ)$ by

$$Φ(x)(t) = \begin{cases} α_i(x_{T_i})(t), & \text{if } t ∈ [i - 1, i] \text{ for some } i ∈ \mathbb{N}, \\ α_{-i}(x_{T_{-i}})(t), & \text{if } t ∈ [-1, -1 + 1] \text{ for some } i ∈ \mathbb{N} \end{cases}$$

for all $x ∈ ⊆ ℝ^ω$ and $t ∈ ℝ$. One can check that for $t$ an integer, the two ways of defining $Φ(x)(t)$ agree. In fact, for each $x ∈ ⊆ ℝ^ω$, $Φ(x)$ is a well-defined function mapping $ℝ$ into $ℝ$ and taking each integer to itself. Also between two consecutive integers, $Φ(x)$ is an
increasing homeomorphism. Therefore, \( \Phi(x) \) is indeed an element of \( \mathcal{H}^+(\mathbb{R}) \).

Let \( H = \Phi(\sqcup \mathbb{R}^\omega) \). Then if \( \mathbb{Z} \) is the set of integers, we have
\[
H = \{ h \in \mathcal{H}_f^+(\mathbb{R}) : h(i) = i \text{ for all } i \in \mathbb{Z} \}.
\]
In order to work with \( \Phi^{-1} \), we give its definition. Define \( \Psi : H \rightarrow \sqcup \mathbb{R}^\omega \) by
\[
\Psi(h)_n = \begin{cases} 
\alpha_i^{-1}(h|_{[i-1,i]})_n, & \text{if } n \in T_i \text{ for some } i \in \mathbb{N}, \\
\alpha_{-i}^{-1}(h|_{[-i,-i+1]})_n, & \text{if } n \in T_{-i} \text{ for some } i \in \mathbb{N},
\end{cases}
\]
for all \( h \in H \) and \( n \in \omega \). It is straightforward to check that \( \Psi \Phi \) is the identity map on \( \sqcup \mathbb{R}^\omega \) and \( \Phi \Psi \) is the identity map on \( H \), so that \( \Phi \) is a bijection from \( \sqcup \mathbb{R}^\omega \) onto the subspace \( H \subseteq \mathcal{H}_f^+(\mathbb{R}) \).

Also it is evident that \( H \) is closed in \( \mathcal{H}_f^+(\mathbb{R}) \). So it remains to show that \( \Phi \) and \( \Psi \) are continuous.

To show that \( \Phi \) is continuous, let \( x \in \sqcup \mathbb{R}^\omega \) and let \( \varepsilon > 0 \). Since \( \alpha_i \) and \( \alpha_{-i} \) are continuous, for each \( i \in \mathbb{N} \) there exist finite subsets \( F_i \) of \( T_i \) and \( F_{-i} \) of \( T_{-i} \) and there exist \( \varepsilon_i \in C_+(T_i) \) and \( \varepsilon_{-i} \in C_+(T_{-i}) \) such that
\[
\alpha_i(B(x_{T_i}, F_i, \varepsilon_i)) \subseteq B(\Phi(x)|_{[i-1,i], \varepsilon_{[i-1,i]}})
\]
and
\[
\alpha_{-i}(B(x_{T_{-i}}, F_{-i}, \varepsilon_{-i})) \subseteq B(\Phi(x)|_{[-i,-i+1], \varepsilon_{[-i,-i+1]}}).
\]
Let
\[
S = \bigcup \{ F_i : i \in \mathbb{N} \} \cup \bigcup \{ F_{-i} : i \in \mathbb{N} \},
\]
which is an element of \( S \). Also let
\[
U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m,
\]
where each
\[
U_m = \begin{cases} 
(x_m - \varepsilon_i(m), x_m + \varepsilon_i(m)), & \text{if } m \in F_i \text{ for some } i \in \mathbb{N}, \\
(x_m - \varepsilon_{-i}(m), x_m + \varepsilon_{-i}(m)), & \text{if } m \in F_{-i} \text{ for some } i \in \mathbb{N}.
\end{cases}
\]
Then \( x \in U \) and
\[
\Phi(U) \subseteq B(\Phi(x), \varepsilon),
\]
showing that \( \Phi \) is continuous.
To show that $\Psi$ is continuous, let $h \in H$ and let

$$U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m$$

be a basic neighborhood of $\Psi(h)$ in $\boxtimes_{\mathbb{R}}^\omega$, where we may assume that each $U_m = (\Psi(h) - \delta_m, \Psi(h) + \delta_m)$ for some $\delta_m > 0$. For each $i \in \mathbb{N}$, since $S \cap T_i$ and $S \cap T_{-i}$ are finite, we can find $\varepsilon_i \in C_+(T_i)$ and $\varepsilon_{-i} \in C_+(T_{-i})$ such that $\varepsilon_i(m) = \delta_m$ for all $m \in S \cap T_i$ and $\varepsilon_{-i}(m) = \delta_m$ for all $m \in S \cap T_{-i}$.

Since $\alpha_i^{-1}$ and $\alpha_{-i}^{-1}$ are continuous, there exist $\sigma_i \in C_+([i - 1, i])$ and $\sigma_{-i} \in C_+([-i, -i + 1])$ such that

$$\alpha_i^{-1}(B(h|_{[i-1,i]}, \varepsilon_i)) \subseteq B(\Psi(h)|_{T_i}, \varepsilon_i)$$

and

$$\alpha_{-i}^{-1}(B(h|_{[-i,-i+1]}, \varepsilon_{-i})) \subseteq B(\Psi(h)|_{T_{-i}}, \varepsilon_{-i}).$$

Let $\sigma \in C_+(\mathbb{R})$ be such that for each $i \in \mathbb{N}$, $\sigma|_{[i-1,i]} \leq \sigma_i$ and $\sigma|_{[-i,-i+1]} \leq \sigma_{-i}$. Then

$$\Psi(B(h, \sigma) \cap H) \subseteq U,$$

showing that $\Psi$ is continuous, and hence that $\Phi$ is a closed embedding.

For the last statement of the theorem, note that for any $S \in \mathcal{S}$, the set

$$\{x \in \boxtimes_{\mathbb{R}}^\omega : x_m = 0 \text{ for all } m \in \omega \setminus S\}$$

is a closed subspace of $\boxtimes_{\mathbb{R}}^\omega$ that is homeomorphic to $\boxtimes_\mathbb{R}^\omega$. □

Although we do not have a proof that $\mathcal{H}_f^+(\mathbb{R})$ is homeomorphic to $\boxtimes_{\mathbb{R}}^\omega$, the second embedding in Theorem 6.5 can be modified to obtain our last theorem. This theorem uses two subspaces of the space $R = \prod_{i \in \mathbb{Z}} \mathbb{R}_i$, where $\mathbb{Z}$ is the set of integers and each $\mathbb{R}_i$ is a copy of $\mathbb{R}$. We say that an element $x$ in $R$ is increasing provided that, for each $i, j \in \mathbb{Z}$ with $i < j$, we have $x_i < x_j$. Also we say that $x$ is unbounded provided that $\lim_{i \to \infty} x_i = \infty$ and $\lim_{i \to -\infty} x_i = -\infty$. The two subspaces used in the next theorem are

$$P = \{x \in R : x \text{ is increasing}\},$$

$$Q = \{x \in R : x \text{ is increasing and unbounded}\}.$$

When $P$ and $Q$ are subspaces of $R$ with the box product topology, we denote them by $\boxtimes P$ and $\boxtimes Q$. 
Theorem 6.6. There exists a homeomorphism \( \Psi \) mapping \( C_f^+(\mathbb{R}) \) onto \( □P \times □ R^\omega \) such that when \( \Psi \) is restricted to \( \mathcal{H}_f^+ (\mathbb{R}) \), it is a homeomorphism from \( \mathcal{H}_f^+ (\mathbb{R}) \) onto \( □Q \times □ R^\omega \).

Proof: For the semi-box product \( □ R^\omega \), we take \( Y = \mathbb{I}, A = \{-1, 1\} \), and \( \phi \) a bijection from \( \omega \) onto a dense subset of \( \mathbb{I} \setminus (A \cup \{1 - 1/i : i \in \mathbb{N}\} \cup \{-1 + 1/i : i \in \mathbb{N}\}) \).

Our construction is a modification of that used in the second embedding of Theorem 6.5. For each \( i \in \mathbb{Z} \), let

\[
T_i = \begin{cases} 
\phi^{-1} \left( [1 - 1/(i + 1), 1 - 1/(i + 2)] \cap \phi(\omega) \right), & \text{if } i \geq 0, \\
\phi^{-1} \left( [-1 - 1/(i - 1), -1 - 1/i] \cap \phi(\omega) \right), & \text{if } i < 0.
\end{cases}
\]

Also for each \( i \in \mathbb{Z} \), by Theorem 2.2, there exists a homeomorphism \( \alpha_i \prod_{m \in T_i} \mathbb{R}_m \to \mathcal{H}_k^+ ([0, 1]) \).

Finally, for each \( i \in \mathbb{Z} \), define \( \beta_i : [i, i + 1] \to [0, 1] \) by \( \beta_i(t) = t - i \).

We start by first defining \( \Phi \), the inverse of \( \Psi \). Define \( \Phi : □P \times □ R^\omega \to C_f^+(\mathbb{R}) \) by

\[
\Phi((x, y))(t) = (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t)) + x_i
\]

for all \( (x, y) \in □P \times □ R^\omega, i \in \mathbb{Z}, \) and \( t \in [i, i + 1] \). To check that \( \Phi((x, y)) \) is well defined, observe that \( \Phi((x, y))(i) = x_i \) for all \( i \in \mathbb{Z} \), and that \( \Phi((x, y)) \) is continuous and increasing between consecutive integers. Since \( x \) is also increasing, this shows that \( \Phi((x, y)) \in C_f^+(\mathbb{R}) \).

Now let us define \( \Psi : C_f^+(\mathbb{R}) \to □P \times □ R^\omega \). For each \( f \in C_f^+(\mathbb{R}) \), define \( \Psi(f) = (x, y) \in □P \times □ R^\omega \) where

\[
x_i = f(i)
\]

for all \( i \in \mathbb{Z} \), and where

\[
y_m = \left[ \alpha^{-1}_i \left( \frac{1}{f(i + 1) - f(i)} f\beta^{-1}_i - \frac{f(i)}{f(i + 1) - f(i)} \right) \right]_m
\]

for all \( i \in \mathbb{Z} \) and \( m \in T_i \). To check that \( \Psi(f) = (x, y) \) is well-defined, let \( i \in \mathbb{Z} \) and \( m \in T_i \). Now \( \beta^{-1}_i \) maps \([0, 1] \) onto \([i, i + 1] \),
and for each \( t \in [i, i + 1] \),
\[
\frac{f(t) - f(i)}{f(i + 1) - f(i)} \in [0, 1]
\]
since \( f \) is increasing. In fact, the function
\[
\frac{1}{f(i + 1) - f(i)} f\beta_i^{-1} - \frac{f(i)}{f(i + 1) - f(i)}
\]
maps \([0, 1]\) onto \([0, 1]\), so that \( y_m \) is well-defined.

To check that \( \Psi \Phi \) is the identity on \( \Box P \times \Box \mathbb{R}^w \), let \( (x, y) \in \Box P \times \Box \mathbb{R}^w \) and let \( (x', y') = \Psi(\Phi((x, y))) \). For each \( i \in \mathbb{Z} \),
\[
\begin{align*}
x'_i &= \Phi((x, y))(i) \\
&= (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(i)) + x_i \\
&= (x_{i+1} - x_i)\alpha_i(y_{T_i})(0) + x_i \\
&= x_i,
\end{align*}
\]
so that \( x' = x \). Also for each \( i \in \mathbb{Z} \) and \( m \in T_i \),
\[
\begin{align*}
y'_m &= \left[ \alpha_i^{-1} \left( \frac{1}{x_{i+1} - x_i} \Phi((x, y))\beta_i^{-1} - \frac{x_i}{x_{i+1} - x_i} \right) \right]_m \\
&= \left[ \alpha_i^{-1} \left( \frac{x_{i+1} - x_i}{x_{i+1} - x_i} \alpha_i(y_{T_i}) + x_i - x_i \right) \right]_m \\
&= \left[ \alpha_i^{-1}(\alpha_i(y_{T_i})) \right]_m \\
&= y_m,
\end{align*}
\]
so that \( y' = y \). This shows that \( \Psi \Phi \) is the identity.

To check that \( \Phi \Psi \) is the identity on \( C_f^+(\mathbb{R}) \), let \( f \in C_f^+(\mathbb{R}) \) and let \( f' = \Phi(\Psi(f)) \). For each \( i \in \mathbb{Z} \) and \( t \in [i, i + 1] \),
\[
\begin{align*}
f'(t) &= (f(i + 1) - f(i))\alpha_i \left( \frac{1}{f(i + 1) - f(i)} f\beta_i^{-1} \\
&\quad - \frac{f(i)}{f(i + 1) - f(i)} \right)(\beta_i(t)) + f(i) \\
&= (f(i + 1) - f(i)) \left( \frac{1}{f(i + 1) - f(i)} f(t) - \frac{f(i)}{f(i + 1) - f(i)} \right) \\
&\quad + f(i) \\
&= f(t).
\end{align*}
\]
It now follows that \( f' = f \), showing that \( \Phi \Psi \) is the identity. So \( \Psi \) is a well-defined bijection from \( C_f^+(\mathbb{R}) \) onto \( \square P \times \square \mathbb{R}^{\omega} \).

Now it should be clear, from the definitions of \( Q, \Phi, \) and \( \Psi \), that when \( \Psi \) is restricted to \( \mathcal{H}_f^+(\mathbb{R}) \), it maps \( \mathcal{H}_f^+(\mathbb{R}) \) onto \( \square Q \times \square \mathbb{R}^{\omega} \).

To show that \( \Psi : C_f^+(\mathbb{R}) \to \square P \times \square \mathbb{R}^{\omega} \) is continuous, let \( f \in C_f^+(\mathbb{R}) \), let \( \langle x, y \rangle = \Psi(f) \), let \( W = \prod_{i \in \mathbb{Z}} W_i \) be a basic neighborhood of \( x \) in \( \square P \), and let

\[
U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m
\]

be a basic neighborhood of \( y \) in \( \square \mathbb{R}^{\omega} \). We may assume that each \( W_i = (x_i - \gamma_i, x_i + \gamma_i) \) for some \( \gamma_i > 0 \), and that each \( U_m = (y_m - \delta_m, y_m + \delta_m) \) for some \( \delta_m > 0 \). For each \( i \in \mathbb{Z} \), since \( S \cap T_i \) is finite, there is an \( \varepsilon_i \in C_+(T_i) \) such that \( \varepsilon_i(m) = \delta_m \) for all \( m \in S \cap T_i \). Also for each \( i \in \mathbb{Z} \), since \( \alpha_i^{-1} \) is continuous, there is a \( \sigma_i \in C_+([0, 1]) \) such that

\[
\alpha_i^{-1}(B(f\beta_i^{-1}, \sigma_i)) \subseteq B(\Psi(f)|_{T_i}, \varepsilon_i).
\]

Finally, let \( \sigma \in C_+(\mathbb{R}) \) be such that for each \( i \in \mathbb{Z} \), \( \sigma|_{[i, i+1]} \leq \sigma_i \beta_i \) and \( \sigma(i) \leq \gamma_i \).

We now check that

\[
\Psi(B(f, \sigma)) \subseteq W \times U.
\]

Let \( f' \in B(f, \sigma) \) and let \( \langle x', y' \rangle = \Psi(f') \). Then for each \( i \in \mathbb{Z} \),

\[
|x'_i - x_i| = |f'(i) - f(i)| < \sigma(i) \leq \gamma_i,
\]

so that \( x'_i \in W_i \). This shows that \( x \in W \). Next let \( i \in \mathbb{Z} \) and \( m \in S \cap T_i \). Since for each \( t \in [i, i+1] \),

\[
|f'(t) - f(t)| < \sigma(t) \leq \sigma_i \beta_i(t),
\]

we have, by equating \( s \) and \( \beta_i(t) \), that for each \( s \in [0, 1] \),

\[
|f'\beta_i^{-1}(s) - f\beta_i^{-1}(s)| < \sigma_i(s).
\]

Therefore,

\[
\Psi(f')|_{T_i} = \alpha_i^{-1}(f'_i, \sigma_i) \subseteq B(\Psi(f)|_{T_i}, \varepsilon_i).
\]

Then for each \( m \in S \cap T_i \),

\[
|y'_m - y_m| < \varepsilon_i(m) = \delta_m.
\]
This shows that each $y_m \in U_m$, and hence $y \in U$. Thus, $\Psi(B(f, \sigma)) \subseteq W \times U$, showing that $\Psi$ is continuous.

To show that $\Phi : \square P \times \square \mathbb{R}^\omega \rightarrow C_f^+(\mathbb{R})$ is continuous, let $(x, y) \in \square P \times \square \mathbb{R}^\omega$ and let $B(\Phi((x, y)), \varepsilon)$ be a basic neighborhood of $\Phi((x, y))$ in $C_f^+(\mathbb{R})$ where $\varepsilon \in C_+(\mathbb{R})$. For each $i \in \mathbb{Z}$, define the following three numbers:

$$
\varepsilon_i = \frac{1}{4} \min\{\varepsilon(t) : t \in [i, i + 1]\}, \\
\delta_i = \frac{\varepsilon_i}{x_{i+1} - x_i}, \\
\gamma_i = \min\{\varepsilon_i, \varepsilon_{i-1}\}.
$$

For each $i \in \mathbb{Z}$, since $\alpha_i^{-1}$ is continuous, there exist a finite subset $F_i$ of $T_i$ and a $\sigma_i > 0$ such that

$$
\alpha_i(B(y_{T_i}, F_i, \sigma_i)) \subseteq B(\alpha_i(y_{T_i}), \delta_i).
$$

Let $S = \cup\{F_i : i \in \mathbb{Z}\}$, which has the property that the set of accumulation points of $\phi(S)$ in $T$ is $A$. Now let $W = \prod_{i \in \mathbb{Z}} W_i$ where $W_i = (x_i - \gamma_i, x_i + \gamma_i)$ for all $i \in \mathbb{Z}$, and let

$$
U = \prod_{m \in S} U_m \times \prod_{m \in \omega \setminus S} \mathbb{R}_m
$$

where $U_m = (y_m - \sigma_i, y_m + \sigma_i)$ for all $i \in \mathbb{Z}$ and $m \in F_i$. Then $W \times U$ is a neighborhood of $(x, y)$ in $\square P \times \square \mathbb{R}^\omega$.

We now check that

$$
\Phi(W \times U) \subseteq B(\Phi((x, y)), \varepsilon).
$$

Let $(x', y') \in W \times U$. Then for each $i \in \mathbb{Z}$ and $t \in [i, i + 1]$, 

$$
\Phi((x', y'))(t) - \Phi((x, y))(t) = [(x'_{i+1} - x_i')\alpha_i(y'_{T_i})(\beta_i(t)) + x_i'] \\
- (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t)) - x_i \\
\leq |x'_{i+1} - x_i'|\alpha_i(y'_{T_i})(\beta_i(t)) - (x_{i+1} - x_i)\alpha_i(y_{T_i})(\beta_i(t)) \\
+ |x_{i+1} - x_i||\alpha_i(y'_{T_i})(\beta_i(t)) - \alpha_i(y_{T_i})(\beta_i(t))| \\
+ |x_i' - x_i|
$$

$$
\leq |x_{i+1} - x_i||\alpha_i(y_{T_i})(\beta_i(t))| + |x_i' - x_i||\alpha_i(y'_{T_i})(\beta_i(t))| \\
+ |x_{i+1} - x_i|\alpha_i(y_{T_i})(\beta_i(t)) - \alpha_i(y_T_i)(\beta_i(t))| + |x_i' - x_i|
$$

$$
\leq (x_{i+1} - x_i) + 2|x_i'| - x_i \\
+ (x_{i+1} - x_i)|\alpha_i(y'_{T_i})(\beta_i(t)) - \alpha_i(y_{T_i})(\beta_i(t))| \\
< \gamma_i + 2\gamma_i + (x_{i+1} - x_i)\delta_i \\
\leq \varepsilon_i + 2\varepsilon_i + \varepsilon_i \\
\leq \varepsilon(t).
$$
Therefore, $\Phi((x',y')) \in B(\Phi((x,y)), \varepsilon)$, so that $\Phi$ is continuous, and thus $\Psi$ is a homeomorphism. \hfill \square

We end by pointing out that it is also possible to map $C^+_T(\mathbb{R})$ onto $\square \mathbb{R}_G$ with a continuous bijection using a construction similar to that in Theorem 6.6, but using the space of positive elements of $R$ instead of the space $P$.

References


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