An Upper Bound for the Cellularity of the Phase Space of a Minimal Dynamical System

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OF A MINIMAL DYNAMICAL SYSTEM

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Abstract. Let $G$ be a topological group acting continuously on an infinite compact space $X$. Suppose the dynamical system $(X, G)$ is minimal. If $G$ is $\kappa$-bounded for some infinite cardinal $\kappa$, then the cellularity of $X$ is at most $\kappa$.

1. Introduction

The purpose of this note is to point out a relation between cardinal invariants of the phase space and the group of a minimal dynamical system.

Generalizing a theorem of Bohuslav Balcar and Alexander Blaszczyk [1], it was shown in [4] that whenever $(G, X)$ is a minimal dynamical system and $G$ is $\aleph_0$-bounded, then the Boolean algebra $\text{ro}(X)$ of regular open subsets of $X$ is the completion of a free Boolean algebra. In particular, $X$ is of countable cellularity. This result is clearly related to an older result of V. V. Uspenski [7], who showed that if an $\aleph_0$-bounded group acts continuously and transitively on a compact space $X$, then $X$ is Dugundji and hence of countable cellularity.

Using some of the ideas from [4], we show that whenever $G$ is a $\kappa$-bounded group and $(G, X)$ is a minimal system, then the cellularity of $X$ is at most $\kappa$.

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This result might be interesting for compact homogeneous spaces. A well-known open question by van Douwen (see [6]) about compact homogeneous spaces is whether the cellularity of such a space can be larger than $2^{\aleph_0}$. One feasible approach to show that it cannot is to try to construct, for a given compact homogeneous space $X$, a $2^{\aleph_0}$-bounded group acting sufficiently transitively on $X$, i.e., in such a way that that $(G, X)$ is a minimal system.

2. Preliminaries

Let $G$ be a topological group and $X$ a compact space. An action of $G$ on $X$ is a homomorphism $\varphi$ from $G$ to the group $\text{Aut}(X)$ of autohomeomorphisms of $X$. The action $\varphi$ is continuous if the map $G \times X \to X; (g, x) \mapsto \varphi(g)(x)$ is continuous. Typically we will not mention $\varphi$ and write $gx$ instead of $\varphi(g)(x)$.

A topological group $G$ together with a topological space $X$ and a continuous action of $G$ on $X$ is a dynamical system. $X$ is the phase space of the system. For every $x \in X$, the set $Gx = \{gx : g \in G\}$ is the $G$-orbit of $x$. The dynamical system $(G, X)$ is minimal if every $G$-orbit is dense in $X$.

For an infinite cardinal $\kappa$, the group $G$ is $\kappa$-bounded if for every non-empty open subset $O$ of $G$ there is a set $S \subseteq G$ of size $\kappa$ such that $SO = G$. Here $SO$ denotes the set $\{gh : g \in S \land h \in O\}$.

The cellularity of $X$ is the least cardinal $\kappa$ such that every family $\mathcal{O}$ of size $> \kappa$ of non-empty open subsets of $X$ contains two distinct sets with a non-empty intersection.

3. Proof of the main result

Let $X$ be a compact space. $C(X)$ denotes the space of continuous real valued functions on $X$ equipped with the sup-norm $\|\|_\infty$. If $G$ acts on $X$ via $\varphi$, then the natural action of $G$ on $C(X)$ is defined by letting $gf = f \circ \varphi(g)$. It is easily checked that $G$ acts on $C(X)$ by isometries and that the action of $G$ on $C(X)$ is continuous if the action on $X$ is continuous.

The action of $G$ on $C(X)$ provides us with a simple way of constructing $G$-equivariant quotients of $X$, i.e., quotients for which the quotient map commutes with the group actions. Let $B$ be a closed
subalgebra of $C(X)$ which is closed under the action of $G$ on $C(X)$. Define an equivalence relation $\sim_B$ on $X$ as follows:

For all $x, y \in X$, let $x \sim_B y$ if and only if for all $b \in B$, $b(x) = b(y)$. It is well known that $X/\sim_B$ is Hausdorff. Since $B$ is closed under the action of $G$, the action of $G$ on $X$ is compatible with $\sim_B$. Hence, there is a natural action of $G$ on $X/\sim_B$. This action is continuous. $X/\sim_B$ is a $G$-equivariant quotient of $X$.

**Definition 3.1.** A continuous map $f : X \to Y$ between topological spaces is semi-open if for every non-empty open set $O \subseteq X$, $f[O]$ has a non-empty interior.

The following is well known.

**Lemma 3.2.** Let $(G, X)$ and $(G, Y)$ be dynamical systems. Assume that $\pi : X \to Y$ is continuous, onto, and $G$-equivariant; i.e., assume that $\pi$ commutes with the actions. Suppose that $(G, X)$ is a minimal system. Then $\pi$ is semi-open.

For the convenience of the reader we include a proof of this lemma.

**Proof:** Suppose $O \subseteq X$ is a non-empty open set. Let $U \subseteq O$ be a non-empty open set with $\operatorname{cl}_X U \subseteq O$. Since $(G, X)$ is minimal, every $G$-orbit in $X$ meets the set $U$. It follows that $GU = X$. Since $X$ is compact, a finite number of translates of $U$ covers $X$. It follows that a finite number of translates of $\pi[U]$ and hence of $\pi[\operatorname{cl}_X U]$ cover $Y$. Since the translates of $\pi[\operatorname{cl}_X U]$ are closed sets, one of them has a non-empty interior, by the Baire Category Theorem. It follows that $\pi[\operatorname{cl}_X U]$, and therefore $\pi[O]$, has a non-empty interior. \qed

**Lemma 3.3.** Let $\kappa$ be an infinite cardinal. Suppose $G$ is a $\kappa$-bounded group acting continuously on a metric space $Z$. Then every $G$-orbit in $Z$ has a dense subset of size $\leq \kappa$.

**Proof:** Let $z \in Z$. For every $n \in \omega$, let $U_n$ be the open ball of radius $\frac{1}{2^n}$ around $z$. Since $G$ acts continuously on $Z$, the map $G \to Z$ defined by $g \mapsto gz$ is continuous. Thus, there is an open neighborhood $V_n$ of the neutral element of $G$ such that $V_n z \subseteq U_n$. Since $G$ is $\kappa$-bounded, there is a set $S_n \subseteq G$ of size $\leq \kappa$ such that $S_n V_n = G$. Now $Gz = S_n V_n z \subseteq SU_n$. It is easily checked that $\bigcup_{n \in \omega} S_n z$ is dense in $Gz$. \qed
In the following, we use elementary submodels of $\mathcal{H}_\chi = (\mathcal{H}_\chi, \in)$ for some infinite cardinal $\chi$. Here, $\mathcal{H}_\chi$ denotes the set of all sets whose transitive closure is of size $< \chi$. Readers not familiar with the method of elementary submodels might consult [2], [3], or [5] for an introduction.

Fix a sufficiently large cardinal $\chi$. Note that, for every cardinal $\kappa$, if $M$ is an elementary submodel of $\mathcal{H}_\chi$ and $\kappa \subseteq M$, then for every set $S \in M$ which is of size $\kappa$, $S \subseteq M$ since $M$ contains a bijection between $\kappa$ and $S$.

**Lemma 3.4.** Let $Z$ be a metric space and suppose that a $\kappa$-bounded group acts continuously on $Z$. If $M$ is an elementary submodel of $\mathcal{H}_\chi$ such that $\kappa \cup \{\kappa, Z, G\} \subseteq M$, then $\text{cl}_Z(Z \cap M)$ is closed under the action of $G$.

**Proof:** Let $z \in Z \cap M$. By Lemma 3.3, $Gz$ has a dense subset $D$ of size $\kappa$. $M$ knows about this and hence we may assume $D \subseteq M$. Since $\kappa \subseteq M$, $D \subseteq M$. It follows that $Gz \subseteq \text{cl}_Z(Z \cap M)$.

Now let $z \in \text{cl}_Z(Z \cap M)$. By the first part of the proof, $G(Z \cap M) \subseteq \text{cl}_Z(Z \cap M)$. Hence,

$$Gz \subseteq G \text{cl}_Z(Z \cap M) = \text{cl}_Z(G(Z \cap M)) \subseteq \text{cl}_Z(Z \cap M).$$

**Corollary 3.5.** Let $(G, X)$ be a dynamical system such that $G$ is $\kappa$-bounded. If $M$ is an elementary submodel of size $\kappa$ of $\mathcal{H}_\chi$ such that $\kappa \cup \{\kappa, X, G\} \subseteq M$, then $B = \text{cl}_C(X \cap M)$ is a closed subalgebra of $C(X)$, which is closed under the action of $G$. In particular, $X/ \sim_B$ is a $G$-equivariant quotient of $X$ of weight $\leq \kappa$.

**Proof:** By Lemma 3.4, $B$ is closed under the action of $G$. It is easily checked that $C(X) \cap M$ is a subalgebra of $C(X)$. It follows that $B = \text{cl}_C(X \cap M)$ is a closed subalgebra of $C(X)$.

Now $X/ \sim_B$ is a $G$-equivariant quotient of $X$. $C(X/ \sim_B)$ is isometrically isomorphic to $B$ and therefore has a dense subset of size $\leq \kappa$. It follows that $X/ \sim_B$ is of weight $\leq \kappa$. □

**Theorem 3.6.** Let $(G, X)$ be a minimal system and suppose that $G$ is $\kappa$-bounded. Then the cellularity of $X$ is at most $\kappa$.

**Proof:** Let $\mathcal{A}$ be a maximal family of pairwise disjoint non-empty open subsets of $X$. Let $M$ be an elementary submodel of $\mathcal{H}_\chi$ of size $\kappa$ such that $\kappa \cup \{\kappa, X, G, \mathcal{A}\} \subseteq M$. Let $B = \text{cl}_C(X \cap M)$. By Corollary 3.5, $X/ \sim_B$ is a $G$-equivariant quotient of $X$ of weight...
\[ \leq \kappa. \] Let \( \pi : X \to X/\sim_B \) be the quotient map. By Lemma 3.2, \( \pi \) is semi-open. Note that \( C(X/\sim_B) \) is isometrically isomorphic to \( B \) via the map
\[ \cdot \circ \pi : C(X/\sim_B) \to B; f \mapsto f \circ \pi. \]

Claim. \( \mathcal{A} \subseteq M. \)

Let \( O \subseteq X \) be non-empty and open. Choose a non-empty open set \( U \subseteq \pi[O]. \) We may assume that \( U \) is of the form \( f^{-1}[\mathbb{R} \setminus \{0\}] \) for some continuous \( f : X/\sim_B \to \mathbb{R} \) with \( f \circ \pi \in \text{cl}_{C(X)}(C(X) \cap M). \)

Choose \( n \in \omega \) so that \( ||f||_\infty - \frac{1}{n} > \frac{1}{n}. \) Let \( f_M : X/\sim_B \to \mathbb{R} \) be such that \( f_M \circ \pi \in C(X) \cap M \) and \( ||f - f_M||_\infty < \frac{1}{n}. \) Now
\[ U_M = f_M^{-1} \left[ \mathbb{R} \setminus \left( \frac{1}{n}, \frac{1}{n} \right) \right] \subseteq U. \]

Note that \( \pi^{-1}[U_M] = (f_M \circ \pi)^{-1} \left[ \mathbb{R} \setminus \left( \frac{1}{n}, \frac{1}{n} \right) \right] \) is an element of \( M \) and a subset of \( O. \)

Since \( M \) knows that \( A \) is a maximal family of disjoint open sets, there is \( A \in \mathcal{A} \cap M \) such that \( A \cap \pi^{-1}[U_M] \) is non-empty. It follows that \( \mathcal{A} \cap M \) is a maximal family of disjoint open subsets of \( X \) and therefore \( \mathcal{A} \subseteq M. \) This finishes the proof of the claim.

Since \( |M| \leq \kappa, |\mathcal{A}| \leq \kappa. \)

\[ \square \]

References


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