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MAKING HOLES IN THE HYPERSPACE OF SUBCONTINUA OF A PEANO CONTINUUM

JOSÉ G. ANAYA

Abstract. Let X be a metric continuum and $C(X)$ the hyperspace of subcontinua of X. Let $A \in C(X)$, A is said to make a hole in $C(X)$, if $C(X) - \{A\}$ is not unicoherent. In this paper we assume that X is locally connected and we characterize those elements $A \in C(X)$ such that A makes a hole in $C(X)$.

1. Introduction

Throughout this paper X will denote a continuum (a nondegenerate compact, connected, metric space) with metric d. A Peano continuum is a locally connected continuum. Let $C(X)$ be the hyperspace of subcontinua of X and let 2^X be the hyperspace of all nonempty closed sets of X. The hyperspaces are considered with the Hausdorff metric H_d. Let Z be a unicoherent topological space and let z be an element of Z. We say that *z makes a hole in Z* if $Z - \{z\}$ is not unicoherent.

We are interested in the following problem:

Problem. Let $\mathcal{H}(X)$ be a hyperspace of X. For which elements $A \in \mathcal{H}(X)$, does A make a hole in $\mathcal{H}(X)$?

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In [1], the author presents some partial solutions to this problem.

In this paper, we present the solution to this problem when X is a Peano continuum and $\mathcal{H}(X) = C(X)$.

More partial answers to the problem posed above are presented in [2].

To finish this section we set some conventions. Denote the unit closed interval $[0, 1]$ by I. An *arc* is any homeomorphic space to I. A *free arc* in a continuum X is an arc pq, where p and q are the end points of pq, such that $pq - \{p, q\}$ is open in X. Let S be a simple closed curve contained in a continuum X. We say that S *is a free simple closed curve in* X if $S \neq X$ and there exists $p \in S$ such that p is not an interior point of S and $S - \{p\}$ is an open subset of X.

2. Auxiliary results

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the set of positive integers and the set of real numbers, respectively. A *map* is a continuous function. A *Whitney map* $\mu : 2^X \to I$ is a map such that $\mu(\{x\}) = 0$, for each $x \in X$, $\mu(X) = 1$ and $\mu(A) < \mu(B)$ whenever $A \subset B \neq A$, the existence of Whitney maps is guaranteed by [7, Theorem 13.4, p. 107].

A map $f : Z \to S^1$, where Z is a topological connected space and S^1 is the unit circle in the Euclidean plane $\mathbb{R}^2$, *has a lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the map of $\mathbb{R}$ on S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z *has property (b)* if each map $f : Z \to S^1$ has a lifting. It is known (see [12, Theorem 7.3, p. 227]) that, if Z is normal and it has property (b), then Z is unicoherent and (see [12, Theorem 7.4, p. 228]) if Z is a locally connected normal T_1-space, then Z is unicoherent if and only if Z has property (b).

We will frequently use the Unique Lifting Theorem (ULT) (see [6, 5.1]) which asserts:

Proposition 2.1. *If Z is a connected space and f, $g : Z \to \mathbb{R}$ are two maps such that* $\exp \circ f = \exp \circ h$ *and* $f(z) = g(z)$*, for some* $z \in Z$*, then* $f = g$*.*

We will also use the following results. The proposition 2.2 is easy to prove and for the proof of Proposition 2.3 use Theorem 2.3 of [5, p. 31].

Proposition 2.2. *Let W be a topological space and let Z and Y be closed connected subsets of W. If Z and Y have property (b) and $Z \cap Y$ is nonempty and connected. Then $Z \cup Y$ has property (b).*

Proposition 2.3. *Let Z be a connected topological space and let Y be a deformation retract of Z. Then Z has property (b) if and only if Y has property (b).*

Proposition 2.4. *Let Z be a connected normal topological space such that Z has property (b), let $p \in Z$ and let W be a neighborhood of p in Z such that $W - \{p\}$ has property (b). If there exists a neighborhood U of p such that bd(U) is connected and cl(U) $\subset W$, then $Z - \{p\}$ has property (b).*

Proof. Let $f : Z - \{p\} \to S^1$ be a map. Since $W - \{p\}$ has property (b), there exists a map $h : W - \{p\} \to \mathbb{R}$ such that $\exp \circ h = f|_{W-\{p\}}$. By Tietze's Extension Theorem, there exists a map $h' : Z \to \mathbb{R}$ such that $h'|_{\mathrm{bd}(U)} = h|_{\mathrm{bd}(U)}$. We may assume that $W \neq Z$, then we can fix a point $z_0 \in \mathrm{bd}(U)$. We define $f' : Z \to S^1$ by

$$f'(x) = \begin{cases} f(x), & \text{if } x \in Z - \mathrm{int}(U), \\ (\exp \circ h')(x), & \text{if } x \in \mathrm{cl}(U). \end{cases}$$

Notice that f' is continuous. Since Z has the property (b), there exists a map, $h_1 : Z \to \mathbb{R}$, such that $\exp \circ h_1 = f'$ and $h_1(z_0) = h(z_0)$. We define $h_2 : Z - \{p\} \to \mathbb{R}$ by

$$h_2(x) = \begin{cases} h_1(x), & \text{if } x \in Z - \mathrm{int}(U), \\ h(x), & \text{if } x \in \mathrm{cl}(U) - \{p\}. \end{cases}$$

Since $h_1|_{\mathrm{bd}(U)}$ and $h|_{\mathrm{bd}(U)}$ are liftings of $f|_{\mathrm{bd}(U)}$, $h_1(z_0) = h(z_0)$ and bd(U) is connected, by Proposition 2.1. □

As a consequence of 4.15 of [12, p. 50] we have:

Proposition 2.5. *Let Z be a unicoherent locally connected continuum and let $p \in Z$ such that p is not a cut point of Z. Then, for each neighborhood W of p, there exists a neighborhood U of p, such that $p \in U \subset W$ and bd(U) is connected.*

If $A \subset X$ and $\varepsilon > 0$, let

$N(\varepsilon, A) = \{x \in X :$ there exists $a \in A$ such that $d(a, x) < \varepsilon\}$

and, if $p \in X$, let

$$B(\varepsilon, p) = \{x \in X : d(x, p) < \varepsilon\}.$$

An *order arc* in 2^X is a nondegenerate subcontinuum β of 2^X such that, if A, $B \in \beta$, then $A \subset B$ or $B \subset A$ (see [9]).

Lemma 2.6. *Let X be a continuum, let $A_0 \in 2^X - \{X\}$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $2^X - \{X\}$ such that $A_0 = \lim A_n$. For each $n \in \mathbb{N} \cup \{0\}$, let α_n be an order arc from A_n to X such that $\alpha_0 = \lim \alpha_n$. Let $f : \mathcal{A} \to S^1$ be a map, where $\mathcal{A} = \bigcup_{n \in \mathbb{N} \cup \{0\}} \alpha_n$, let $t_0 \in \exp^{-1}(f(X))$ and, for each $n \in \mathbb{N} \cup \{0\}$, let $h_n : \alpha_n \to \mathbb{R}$ be a lifting of $f|_{\alpha_n}$ such that $h_n(X) = t_0$. Then $h_0(A_0) = \lim h_n(A_n)$.*

Proof. Let $\mu : 2^X \to [0,1]$ be a Whitney map. For each $n \in \mathbb{N} \cup \{0\}$, μ_n will denote the function $\mu|_{\alpha_n} : \alpha_n \to \mu(\alpha_n)$.

Clearly, for each $n \in \mathbb{N} \cup \{0\}$, μ_n is a homeomorphism.

For each $n \in \mathbb{N} \cup \{0\}$, we define the function $f_n : [0,1] \to [\mu(A_n), 1]$ by $f_n(t) = (\mu_n(A_n) - 1)t + 1$. Clearly, for each $n \in \mathbb{N} \cup \{0\}$, f_n is a homeomorphism.

For each $i \in \mathbb{N}$, let L_i be the convex segment, contained in the Euclidean plane $\mathbb{R}^2$ which joins the points $(0,0)$ and $\left(1, \frac{1}{i}\right)$. Let $L_0 = [0,1] \times \{0\}$ and let $Y = \bigcup_{i=0}^{\infty} L_i$. We denote by v the point $(0,0)$.

For each $n \in \mathbb{N} \cup \{0\}$, we define the function $p_n : L_n \to [0,1]$ by $p_n(x,y) = x$. Notice that, p_n is a homeomorphism.

Since, for each $n \in \mathbb{N} \cup \{0\}$, μ_n, f_n and p_n are homeomorphisms, then $g_n = (\mu_n)^{-1} \circ f_n \circ p_n$ is a homeomorphism between L_n and α_n. We define $g : Y \to \mathcal{A}$ by $g(x,y) = g_n(x,y)$, if $(x,y) \in L_n$, where $n \in \mathbb{N} \cup \{0\}$. Notice that, for each $n \in \mathbb{N} \cup \{0\}$, $g_n(v) = X$. Hence g is well defined.

Claim 1. g is continuous.

Since each g_n is continuous, we only need to check the continuity of g at the point $(x_0, y_0) \in L_0$. Take a sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ in Y such that $\lim (x_k, y_k) = (x_0, y_0)$. Since $\mathcal{A}$ is compact, we may assume that there exists $B \in \mathcal{A}$ such that $B = \lim g(x_k, y_k)$. We need to show that $B = g(x_0, y_0)$. We may also assume that, for each $k \in \mathbb{N}$, $(x_k, y_k) \in L_{n_k}$, for some $n_k \geq 1$ and $n_1 < n_2 < \ldots$.

Since $\lim \alpha_{n_k} = \alpha_0$ and $g(x_k, y_k) \in \alpha_{n_k}$, for each $k \in \mathbb{N}$, we have $B \in \alpha_0$. Since $g(x_0, y_0) \in \alpha_0$ and α_0 is an order arc, $g(x_0, y_0) \subset B$ or $B \subset g(x_0, y_0)$. Notice that $\mu(g(x_0, y_0)) = (\mu(A_0) - 1)x_0 + 1$ and

$$\mu(B) = \lim \mu(g(x_k, y_k)) = \lim (\mu(A_{n_k}) - 1) x_k + 1 = (\mu(A_0) - 1) x_0 + 1 = \mu(g(x_0, y_0)).$$

Thus $\mu(B) = \mu(g(x_0, y_0))$ and $g(x_0, y_0) \subset B$ or $B \subset g(x_0, y_0)$. Hence $B = g(x_0, y_0)$. We have proved that g is continuous.

Since Y is contractible, by Proposition 2.3, Y has property (b). Since, $f \circ g : Y \to S^1$ is a map, there exists a map $l : Y \to \mathbb{R}$, such that $f \circ g = \exp \circ l$. Moreover, since $g(v) = X$ and $f(g(v)) = f(X)$, we can assume that $l(v) = t_0$.

Given $n \in \mathbb{N} \cup \{0\}$, notice that $l|_{L_n}$ is a lifting of $f \circ g|_{L_n}$, such that $l|_{L_n}(v) = t_0$. Also notice that $\exp \circ (h_n \circ g)|_{L_n} = f \circ g|_{L_n}$, and $(h_n \circ g)|_{L_n}(v) = h_n(X) = t_0$.

Therefore, $l|_{L_n}$ and $h_n \circ g|_{L_n}$ are liftings of $f \circ g|_{L_n}$. Moreover $l|_{L_n}(v) = h_n \circ g|_{L_n}(v)$. Thus, $l|_{L_n} = h_n \circ g|_{L_n}$ (see Proposition 2.1).

Let $q_0 = (1, 0)$ and, for each $n \in \mathbb{N}$, let $q_n = \left(1, \frac{1}{n}\right)$. Notice that, for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} l(q_n) &= (h_n \circ g|_{L_n})(q_n) \\ &= h_n\left(\left(\mu_n^{-1} \circ f_n \circ p_n\right)(q_n)\right) \\ &= h_n\left(\mu_n^{-1}\left((\mu(A_n) - 1)1 + 1\right)\right) \\ &= h_n(A_n). \end{aligned}$$

Since $q_0 = \lim q_n$ and l is a map, we have $l(q_0) = \lim l(q_n)$. Thus, $h_0(A_0) = \lim h_n(A_n)$. This finishes the proof of the lemma. □

3. Locally connected subcontinua

In this section we prove that if X is a Peano continuum and A is a locally connected subcontinuum of X, such that, A is neither a free arc nor a free simple closed curve, then A does not make a hole in $C(X)$. The next lemma is easy to prove.

Lemma 3.1. *Let X be a Peano continuum and let $A_0 \in C(X)$. Then, for each $A \in C(X)$, we have $A \in bd_{C(X)}(C(A_0))$ if and only if $A \subset A_0$ and $A \cap bd_X(A_0) \neq \emptyset$.*

Proposition 3.2. *Let X be a Peano continuum and let A_0 be a locally connected subcontinuum of X such that A_0 is not a free arc. Then*

$$bd_{C(X)}(C(A_0)) - \{A_0\}$$

is pathwise connected.

Proof. We denote the set $\text{bd}_{C(X)}(C(A_0)) - \{A_0\}$ by $\mathcal{D}$.

Let $A_1, A_2 \in \mathcal{D}$. By Lemma 3.1, there exist $a_1 \in \text{bd}_X(A_0) \cap A_1$ and $a_2 \in \text{bd}_X(A_0) \cap A_2$.

Case 1. $a_1 = a_2$.

By Theorem 1.8 of [9, p. 59], there exists a map $h : [0,1] \to C(X)$ such that $h(0) = \{a_1\}$, $h(1) = A_1$ and, for any $s, t \in [0,1]$, $s < t$, we have $h(s) \subset h(t)$. Notice that, for each $t \in [0,1]$, $h(t) \cap \text{bd}_X(A_0) \neq \emptyset$. Moreover, for each $t \in [0,1]$, $h(t)$ is a proper subcontinuum of A_0. By Lemma 3.1, for each $t \in [0,1]$, $h(t) \in \mathcal{D}$. Thus $\{a_1\}$ and A_1 can be connected by a path in $\mathcal{D}$. Similarly $\{a_1\}$ and A_2 (and so A_1 and A_2) can be connected by a path in $\mathcal{D}$.

Case 2. $a_1 \neq a_2$.

Since A_0 is arcwise connected, there exists an embedding $h : [0,1] \to A_0$ such that $h(0) = a_1$ and $h(1) = a_2$. Notice that $h([0,1]) \in C(A_0)$. Since $a_1 \in \text{bd}_X(A_0) \cap h([0,1])$, by Lemma 3.1, $h([0,1]) \in \text{bd}_{C(X)}(C(A_0))$.

Subcase 2.1. $h([0,1]) \neq A_0$.

By the Case 1, A_1 and $h([0,1])$, as well as A_2 and $h([0,1])$ can be connected by a path in $\mathcal{D}$. Thus A_1 and A_2 can be connected by a path in $\mathcal{D}$.

Subcase 2.2. $h([0,1]) = A_0$.

In this case A_0 is an arc and its end points are $h(0)$ and $h(1)$. Since A_0 is not a free arc, $A_0 - \{h(0), h(1)\}$ is not open in X. Therefore there exists $y_0 \in A_0 - \{h(0), h(1)\}$ such that $y_0 \notin \text{int}(A_0 - \{h(0), h(1)\})$. Then $y_0 \notin \text{int}_X(A_0)$. We consider the subarcs $h(0)y_0$ and $y_0h(1)$ of $h([0,1])$. By Case 1, each one of the following pairs of elements of $\mathcal{D}$ can be joined by a path in $\mathcal{D}$: $h(0)y_0$ and $y_0h(1)$; $h(0)y_0$ and A_1; $y_0h(1)$ and A_2. Hence A_1 and A_2 can be joined by a path in $\mathcal{D}$. This completes the proof that $\mathcal{D}$ is pathwise connected. □

A metric d for a continuum X is said to be *convex* provided that given $x, y \in X$, there exists $z \in X$ such that $d(x,z) = d(x,y)/2 = d(z,y)$. By [3] and [8], each Peano continuum admits a convex metric.

We consider $K_d : [0,\infty) \times 2^X \to 2^X$, defined by $K_d(t,A) = \bigcup_{a \in A} C_t^d(a)$, where $C_t^d(a) = \{x \in X : d(x,a) \leq t\}$. If d is a convex metric in X, K_d is a map (see, e.g., [11, p. 172]). Observe also that if $A \in C(X)$, then $K_d(t,A) \in C(X)$.

Lemma 3.3. *Let X be a Peano continuum with convex metric d and let A_0 be a proper subcontinuum of X.*

a) If $B_0 \in C(X) - \{X\}$, then $\alpha = \{K_d(t, B_0) : t \in [0, \infty)\}$ is an order arc from B_0 to X in $C(X)$; moreover, if $B_0 \in bd_{C(X)}(C(A_0))$, then α satisfies $C(A_0) \cap \alpha = \{B_0\}$.

b)If a sequence B_n in $C(X)$ converges to B_0, then $\beta_0 = \lim \beta_n$, where, $\beta_n = \{K_d(t, B_n) : t \in [0, \infty)\}$ for each $n \in \mathbb{N} \cup \{0\}$.

Proof. Obviously, α is an order arc from $B_0 = K_d(0, B_0)$ to $X = K_d(r_0, B_0)$, where $r_0 = diamX$.
Now suppose that $B_0 \in \mathrm{bd}_{C(X)}(C(A_0))$. Let $A \in \alpha \cap C(A_0)$ and $A = K_d(t_0, B_0)$ for some $t_0 \in (0, \infty)$. By Lemma 3.1, there exists a point $b_0 \in B_0 \cap \mathrm{bd}_X(A_0)$. Then $B^d_{t_0}(b_0) \subset K_d(t_0, B_0) = A$ and $B^d_{t_0}(b_0) \cap (X - A_0) \neq \emptyset$. This contradicts the fact that $A \in C(A_0)$.

Part b) follows easily by the continuity. □

Lemma 3.4. *Let X be a Peano continuum, let $A_0 \in C(X) - \{X\}$ and let $C_0 \in bd_{C(X)}(C(A_0))$. Then $cl_{C(X)}(C(X) - C(A_0)) - \{C_0\}$ has property (b).*

Proof. Let $\mathcal{D} = \mathrm{cl}_{C(X)}(C(X) - C(A_0)) - \{C_0\}$, $\mathcal{C} = C(X) - C(A_0)$ and $\mathcal{F} = \mathrm{bd}_{C(X)}(C(A_0)) - \{C_0\}$.

Notice that $\mathcal{D} = \mathcal{C} \cup \mathcal{F}$.

Since $X \in \mathcal{C}$ and if α is an order arc in $C(X)$ from $A \in \mathcal{C}$ to X, we have $\alpha \subset \mathcal{C}$. By Lemma 13 of [1, p. 2004], $\mathcal{C}$ has property (b).

Let $f : \mathcal{D} \to S^1$ be a map. We prove that there exists a map, $h : \mathcal{D} \to \mathbb{R}$ such that $f = \exp \circ h$.

Fix $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(X)$.

Since $\mathcal{C}$ has property b), there exists a map $h_1 : \mathcal{C} \to \mathbb{R}$ such that $\exp \circ h_1 = f|_{\mathcal{C}}$ and $h_1(X) = t_0$.

Since X is locally connected, X admits a convex metric ρ. By (a) of Lemma 3.3, for each $A \in \mathcal{F}$,

$$\alpha_A = \{K(t, A) : t \in [0, \infty)\}$$

is an order arc from A to X in $C(X)$ such that $\alpha_A \cap C(A_0) = \{A\}$. This implies that $\alpha_A \cap \mathcal{F} = \{A\}$. Hence $C_0 \notin \alpha_A$. Moreover, $\alpha_A - \{A\} \subset C(X) - C(A_0)$. Hence $\alpha_A \subset \mathcal{C} \cup \mathcal{F} = \mathcal{D}$. Since α_A is an arc, α_A has property (b). Then, there exists a map, $h_A : \alpha_A \to \mathbb{R}$, such that $f|_{\alpha_A} = \exp \circ h_A$ and $h_A(X) = t_0$.

We define $h : \mathcal{D} \to \mathbb{R}$ by

$$h(A) = \begin{cases} h_A(A), & \text{if } A \in \mathcal{F}, \\ h_1(A), & \text{if } A \in \mathcal{C}. \end{cases}$$

Clearly h is well defined. We are going to prove that:
i) h is continuous and
ii) $f = \exp \circ h$.

i). Since $C(X) - C(A_0)$ is open in $C(X)$, $\mathcal{C}$ is open in $\mathcal{D}$. Moreover $h|_{\mathcal{C}} = h_1$ is continuous. Thus h is continuous at each element of $\mathcal{C}$.

Now we consider $B_0 \in \mathcal{F}$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}$ such that $B_0 = \lim B_n$.

Clearly, for each $n \in \mathbb{N} \cup \{0\}$, $\beta_n = \{K_\rho(t, B_n) : t \in [0, \infty)\} \subset \mathcal{D}$. By Lemma 3.3, $\lim \beta_n = \beta_0$. Given $n \in \mathbb{N}$, if $B_n \in \mathcal{C}$, then $h_1|_{\beta_n}$ is a lifting of $f|_{\beta_n}$ such that $h_1|_{\beta_n}(X) = t_0$. Thus, for each $n \in \mathbb{N} \cup \{0\}$, $h|_{\beta_n}$ is a lifting of $f|_{\beta_n}$ such that $h|_{\beta_n}(X) = t_0$. By Lemma 2.6, h is continuous at B_0.

Clearly, for each $A \in \mathcal{D}$, $f(A) = \exp \circ h(A)$. □

Proposition 3.5. *Let X be a Peano continuum and let A_0 be a proper locally connected subcontinuum of X such that A_0 is neither a simple closed curve nor a free arc. Then A_0 does not make a hole in $C(X)$.*

Proof. Since A_0 is not a simple closed curve, by Theorem 2 of [1, p. 2001], A_0 does not make a hole in $C(A_0)$. Therefore $C(A_0) - \{A_0\}$ is unicoherent. By Theorem 1.92 of [9, p. 134], $C(A_0) - \{A_0\}$ is locally connected. Thus $C(A_0) - \{A_0\}$ has property (b) (see [12, Theorem 7.4, p. 228]).

Since A_0 is a proper subcontinuum of X, $\text{bd}_X(A_0) \neq \emptyset$ and $\text{bd}_X(A_0) \subset A_0$. By Lemma 3.1, $A_0 \in \text{bd}_{C(X)}(C(A_0))$. Hence, by Lemma 3.4, $\text{cl}_{C(X)}(C(X) - C(A_0)) - \{A_0\}$ has property (b).

Notice that

$$(C(A_0) - \{A_0\}) \cap \left(cl_{C(X)}(C(X) - C(A_0)) - \{A_0\}\right) = \text{bd}_{C(X)}(C(A_0)) - \{A_0\}$$

and, by Proposition 3.2, $\text{bd}_{C(X)}(C(A_0)) - \{A_0\}$ is connected. Thus $(C(A_0) - \{A_0\}) \cap \left(cl_{C(X)}(C(X) - C(A_0)) - \{A_0\}\right)$ is connected.

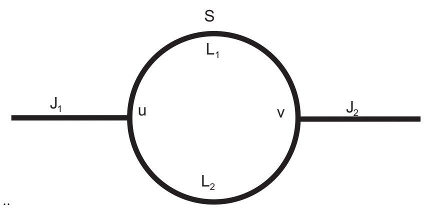

FIGURE 1. Graph G_0.

By Proposition 2.2,

$$C(X) - \{A_0\} = \\ (C(A_0) - \{A_0\}) \cup \left(cl_{C(X)}(C(X) - C(A_0)) - \{A_0\}\right)$$

has property (b). Therefore, $C(X) - \{A_0\}$ is unicoherent. □

4. Finite graphs

In this section we prove that if G_0 is the finite graph of Figure 1, then the simple closed curve S of G_0 does not make a hole in $C(G_0)$.

The edges of G_0 are J_1, J_2, L_1 and L_2. Notice that, each edge of G_0 can be identified with the interval $[0,1]$. We assume that the metric d in G_0 is the metric of arc length and each edge of G_0 has length equal to one.

In this section, if $a, b \in G_0$ and $a \neq b$, ab will denote an arc joining a and b in G_0. We also denote $aa = \{a\}$ for each $a \in G_0$.

We define the sets:

$$\mathcal{M}(\{u\}) = \{ua_1 \cup ub_1 \cup ub_2 : ua_1 \subset J_1,\ ub_1 \subset L_1 \text{ and } ub_2 \subset L_2\},$$
$$\mathcal{M}(\{v\}) = \{va_2 \cup vb_1 \cup vb_2 : va_2 \subset J_2,\ vb_1 \subset L_1 \text{ and } vb_2 \subset L_2\},$$
$$\mathcal{M}(L_1) = \{L_1 \cup ua_1 \cup va_2 \cup ub_2 \cup vc_2 : ua_1 \subset J_1,\ va_2 \subset J_2 \text{ and } \\ ub_2 \cup vc_2 \subset L_2\}, \text{ and}$$
$$\mathcal{M}(L_2) = \{L_2 \cup ua_1 \cup va_2 \cup ub_1 \cup vc_1 : ua_1 \subset J_1,\ va_2 \subset J_2 \text{ and } \\ ub_1 \cup vc_1 \subset L_1\}.$$

By Lemma 5.2 of [4, p. 271] $\mathcal{M}(\{u\})$ and $\mathcal{M}(\{v\})$ are 3-cells, $\mathcal{M}(L_1)$ and $\mathcal{M}(L_2)$ are 4-cells and by [4, Theorem 5.4, p. 272] we have,

$$C(G_0) = \mathcal{M}(\{u\}) \cup \mathcal{M}(\{v\}) \cup \mathcal{M}(L_1) \cup \mathcal{M}(L_2) \cup C(J_1) \cup \\ C(J_2) \cup C(L_1) \cup C(L_2).$$

Clearly $S \in \mathcal{M}(\{u\}) \cap \mathcal{M}(\{v\}) \cap \mathcal{M}(L_1) \cap \mathcal{M}(L_2)$.

Proposition 4.1. *S does not make a hole in $C(G_0)$.*

Proof. It follows easily from Propositon 2.2 that

$$(\mathcal{M}(\{u\}) \cup \mathcal{M}(L_1) \cup \mathcal{M}(L_2) \cup \mathcal{M}(\{v\})) - \{S\},$$

has property b).

By Proposition 2.5, there exists a neighborhood $\mathcal{U}$ of S with connected boundary such that

$$S \in \mathcal{U} \subset \mathrm{cl}(\mathcal{U}) \subset \mathcal{M}(\{u\}) \cup \mathcal{M}(\{v\}) \cup \mathcal{M}(L_1) \cup \mathcal{M}(L_2).$$

Then, by Proposition 2.4, $C(G_0) - \{S\}$ has property (b). □

5. Simple closed curves

In this section we prove that if X is a Peano continuum and S is a simple closed curve in X, then S makes a hole in $C(X)$ if and only if S is a free simple closed curve in X.

Throughout this section G_0 will denote the graph defined in Section 4.

Lemma 5.1. *Let X be a Peano continuum with metric d and let S be a simple closed curve in X such that S is a proper subset of X and S is not a free simple closed curve. Then there exists a subcontinuum G of X such that G is homeomorphic to G_0 and $S \subset G$.*

Proof. Since S is a proper subset of X, $\mathrm{int}(S) \subsetneq S$. Since S is not a free simple closed curve, $S - \mathrm{int}(S)$ is not degenerate. Let K_1, K_2 be disjoint open connected (and then arcwise connected [10, Theorem 8.26, p. 132]) subsets of X such that $q_1 \in K_1$ and $q_2 \in K_2$. For each $i \in \{1, 2\}$, let $p_i \in K_i - S$. Then there exists $f_i : [0, 1] \to X$ a one-to-one map such that $f_i(0) = p_i$ and $f_i(1) = q_i$. Let $t_i = \inf\{t \in [0, 1] : f_i(t) \in S\}$. Clearly $f(t_i) \in S$. Notice that $f_i([0, t_i])$ is an arc and $f_i([0, t_i]) \cap S = \{f_i(t_i)\}$. Notice also that $f_1([0, t_1]) \cap f_2([0, t_2]) = \emptyset$.

Define $G = f_1([0, t_1]) \cup S \cup f_2([0, t_2])$. Hence G is homeomorphic to G_0. □

Lemma 5.2. *Let X be a Peano continuum. Suppose that G_0 is a subcontinuum of X and S is the simple closed curve in G_0. Then $bd_{C(X)}(C(G_0)) - \{S\}$ is connected.*

Proof. If $S \notin \mathrm{bd}_{C(X)}(C(G_0))$, then, since $\mathrm{cl}(C(X) - C(G_0))$ and $C(G_0)$ are connected closed subsets of $C(X)$ such that $C(X) = C(G_0) \cup \mathrm{cl}(C(X) - C(G_0))$ and $C(X)$ is unicoherent (see [7, Theorem 19.8, p. 159] or [1, Lemma 13, p. 2004]), we can conclude $\mathrm{bd}_{C(X)}(C(G_0)) - \{S\} = \mathrm{bd}_{C(X)}(C(G_0))$ is connected.

Now we suppose that $S \in \mathrm{bd}_{C(X)}(C(G_0))$. Let A be an element of $\mathrm{bd}_{C(X)}(C(G_0) - \{S\})$. By Lemma 3.1, $A \cap \mathrm{bd}_X(G_0) \neq \emptyset$. We are going to construct an order arc α from A to G_0 such that $S \notin \alpha$. By Lemma 3.1, we will have that $\alpha \subset \mathrm{bd}_{C(X)}(C(G_0)) - \{S\}$. This will finish the proof of the lemma. If $A \nsubseteq S$, simply take any order arc α from A to G_0. Thus, we may assume that $A \subseteq S$. If $u \in A$, take an order arc from A to $A \cup J_1$ and an order arc α_2 from $A \cup J_1$ to G_0, then $\alpha = \alpha_1 \cup \alpha_2$ is an order arc from A to G_0. So suppose $u \notin A$. Similarly, we assume that $v \notin A$. Then $A \subset L_i$, for some $i \in \{1, 2\}$. In this case take an order arc from A to L_i and now we are in the case we have solved. □

Proposition 5.3. *Let X be a Peano continuum and let S be a simple closed curve contained in X. If S is not a free simple closed curve in X and $S \neq X$, then S does not make a hole in $C(X)$.*

Proof. By Lemma 5.1, we can assume that $S \subset G_0 \subset X$. Notice that $C(G_0) - \{S\}$ and $\mathrm{cl}(C(X) - C(G_0)) - \{S\}$ have property (b) (see Proposition 4.1 and Lemma 3.4, respectively). By Lemma 5.2,

$$\begin{gathered}\mathrm{bd}_{C(X)}(C(G_0)) - \{S\} = \\ (C(G_0) - \{S\}) \cap (\mathrm{cl}(C(X) - C(G_0)) - \{S\}),\end{gathered}$$

is connected. Thus, by Lemma 2.2, $C(X) - \{S\} = (C(G_0) - \{S\}) \cup (cl(C(X) - C(G_0)) - \{S\})$, has property (b). □

Finally we present the main result of this section.

Proposition 5.4. *Let X be a Peano continuum and let S be a simple closed curve such that S is a proper subset of X. Then S makes a hole in $C(X)$ if and only if S is a free simple closed curve.*

Proof. If S is not a free simple closed curve, by Propositon 5.3, S does not make a hole in $C(X)$. If S is a free simple closed curve, by Theorem 5 of [1, p. 2001], S makes a hole in $C(X)$. □

6. Non-locally connected subcontinua

In this section we prove that if X is a Peano continuum and if A is a subcontinuum of X such that A is not locally connected, then A does not make a hole in $C(X)$.

Let X be a Peano continuum and let $A \in C(X)$ be such that A is not locally connected. Denote

$$F(A) = \{a \in A : A \text{ is not locally connected at } a\}.$$

The next lemma is not difficult to prove.

Lemma 6.1. *Let X be a Peano continuum with convex metric d, let $t_0 \in (0,\infty)$ and let B be a subcontinuum of X. If $A = K_d(t_0, B)$, A is locally connected.*

Proposition 6.2. *Let X be a Peano continuum and let $A \in C(X)$ be such that A is not locally connected. Then A does not make a hole in $C(X)$.*

Proof. Let $f : C(X) - \{A\} \to S^1$ be a map. Fix $t_0 \in \exp^{-1}(f(X))$.

Since X is a Peano continuum, X admits a convex metric ρ.

By Lemmas 3.3 and 6.1, $\alpha_B = \{K_\rho(t, B) : t \in [0,\infty)\}$ is an order arc from B to X in $C(X) - \{A\}$, for each $B \in C(X) - \{A, X\}$. Let $h_B : \alpha_B \to \mathbb{R}$ be a map such that $f|_{\alpha_B} = \exp \circ h_B$ and $h_B(X) = t_0$.

We define $h : C(X) - \{A\} \to \mathbb{R}$ by

$$h(B) = \begin{cases} h_B(B), & \text{if } B \neq X, \\ t_0, & \text{if } B = X. \end{cases}$$

Claim 1. h is a map.

First we prove that h is continuous at X. Let $\varepsilon > 0$ be such that $2\varepsilon < 1$. Let $\delta > 0$ be such that $f(B(\delta, X)) \subset \exp((t_0 - \varepsilon, t_0 + \varepsilon))$. Then

$$h' = \left(\left(\exp|_{[t_0-\varepsilon, t_0+\varepsilon]} \right)^{-1} \circ f|_{B(\delta,X)} \right),$$

is a lifting of $f|_{B(\delta,X)}$. If $B \in B(\varepsilon, X)$, then $h'|_{\alpha_B}$ is a lifting of $f|_{\alpha_B}$.

Since $h'|_{\alpha_B}(X) = t_0 = h_B(X)$, by Proposition 2.1, $h'|_{\alpha_B} = h_B$. Hence $h_B(B) \in [t_0 - \varepsilon, t_0 + \varepsilon]$.

Let $B_0 \in C(X) - \{A, X\}$ and let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $C(X) - \{A, X\}$ such that $B_0 = \lim B_n$. By Lemmas 2.6 and 3.3, h is continuous at B_0.

Hence A does not make a hole in $C(X)$. □

7. Main Result

Theorem 7.1. *Let X be a locally continuum and let A be a subcontinuum of X. Then A makes a hole in $C(X)$ if and only if $A = X$ and X is a simple closed curve or A is either a free simple closed curve or a free arc with end points p, q such that $p, q \notin int(A)$.*

Proof. (Necessity) Suppose that A makes a hole in $C(X)$. If $A = X$, by Theorem 2 of [1, p. 2001] X is a simple closed curve. So, we may assume that $A \neq X$. By Proposition 6.2, A is locally connected. By Proposition 3.5, A is either a simple closed curve or a free arc. In the case that A is a simple closed curve, by Propositon 5.4, A is a free simple closed curve. In the case that A is a free arc with end points p and q, by Theorem 4 of [1, p. 2001], $p, q \notin int(A)$. The sufficiency follows from Theorems 1 and 5 of [1]. □

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