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by

Keiko Chiba

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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EXPANDABILITIES OF PRODUCT SPACES

KEIKO CHIBA

Abstract. 1. Suppose X is a normal P-space and Y is a paracompact Σ-space. Then the following are hold.
(1) If X is σ-expandable, then so is $X \times Y$.
(2) If X is θ-expandable, then so is $X \times Y$.
2. Suppose X is a normal P-space and Y is a metrizable space. Then the following are hold.
(3) If X is discretely σ-expandable, then so is $X \times Y$.
(4) If X is discretely θ-expandable, then so is $X \times Y$.
(5) If X is discretely subexpandable, then so is $X \times Y$.
(6) If X is subexpandable, then so is $X \times Y$.

1. Introduction

Throughout this paper we assume that each space is a T_1-space and each map is assumed to be continuous, and it is assumed that any indexed collection is a one-to-one indexing. That is, if a collection $\{G_\xi \mid \xi \in \Xi\}$ is given then $G_\xi \neq G_\eta$ whenever $\xi \neq \eta$.

A space X is called *expandable* (resp. *discretely expandable*) if for every locally finite (resp. discrete) collection $\{F_\xi \mid \xi \in \Xi\}$ of subsets in X, there exists a locally finite collection $\{G_\xi \mid \xi \in \Xi\}$ of open subsets in X such that $F_\xi \subset G_\xi$ for each ξ.

In 1958, M. Katětov [6] proved that in a normal space X, X is expandable if and only if X is collectionwise normal and countably paracompact. Morita [9] introduced the notion of P-spaces. Collectionwise normality of product spaces of P-spaces with metrizable spaces, the following theorem was proved.

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(I) ([9],[11]). Suppose X is a normal P-space and Y is a metrizable space. If X is collectionwise normal, then $X \times Y$ is collectionwise normal and countably paracompact.

It is well known that each normal P-space is countably paracompact. Thus, we have

Theorem A. Suppose X is a normal P-space and Y is a metrizable space. If X is expandable, then so is $X \times Y$.

Morita has introduced a classes of spaces which he terms M-spaces. He gave the following characterization.

(II) ([9]). A space X is an M-space (resp. a paracompact M-space) if and only if there is a quasi-perfect map (resp. a perfect map) from X onto a metrizable space Y.

Theorem A is easily generalized as follows:

Theorem 1.1. *Suppose X is a normal P-space and Y is a paracompact M-space. If X is expandable, then so is $X \times Y$.*

Proof. By (II), there are a metrizable space T and a perfect map $f : Y \to T$. Let $g = 1_X \times f : X \times Y \to X \times T$. Then g is a perfect map. By Theorem A, $X \times T$ is expandable. By Theorem 3.4([8]), $X \times Y$ is expandable. □

In normal spaces, collectionwise normality is equivalent to discrete expandability (12]). Thus we have

Theorem B. Suppose X is a normal P-space and Y is a metrizable space. If X is discretely expandable, then so is $X \times Y$.

In [7], Katuta introduced new notions of θ-expandability, subexpandability, etc., and he has obtained various results concerning these notions.

In this paper we shall prove similar theorems to Theorem A hold for σ-expandability, θ-expandability, discrete σ-expandability, discrete θ-expandability, discrete subexpandability and subexpandability. For σ-expandability and θ-expandability, our theorems are of a more general form. We shall investigate paracompact Σ-spaces. Σ-spaces are introduced by Nagami ([10]). It is known that metrizable spaces are paracompact M-spaces and M-spaces are Σ-spaces.

Let Ω be a set. Denote $\Omega^n = \{(\alpha_0, \alpha_1, ..., \alpha_{n-1}) | \alpha_i \in \Omega, i = 0, ..., n-1\}$ for each $n \in \omega, \Omega^{<\omega} = \bigcup_{n\in\omega} \Omega^n$ and $\Omega^\omega = \{(\alpha_0, \alpha_1, ..., \alpha_n, ...,) | \alpha_n \in \Omega$ for each $n \in \omega\}$. For each $\sigma = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in \Omega^n$ and $\alpha \in \Omega$, we denote $\sigma \vee \alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}, \alpha)$. For each $\sigma = (\alpha_0, \alpha_1, ..., \alpha_n, ...) \in \Omega^\omega$, we denote $\sigma \upharpoonright n = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$. It is obvious that $\sigma \upharpoonright n \in \Omega^n$.

A space X is said to be a P-space ([9]) if for any open cover $\{U(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of X where $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$ and $\alpha \in \Omega$, then there is a closed cover $\{K(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of X such that

(P_1) $K(\sigma) \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,

(P_2) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} U(\sigma \upharpoonright n) = X$, then $\bigcup_{n\in\omega} K(\sigma \upharpoonright n) = X$.

A space X is called a Σ-space if X has a Σ-net.

Lemma 1.2. *([10]). If X is a Σ-space, then X has a spectral Σ-net $\mathcal{F}$, i.e., satisfying the following conditions:*

(i) $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{F}_n$ is a locally finite closed cover of X,

(ii) $\mathcal{F}_n = \{F(\sigma) \mid \sigma \in \Omega^n\}, F(\sigma) = \bigcup_{\alpha\in\Omega} F(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{F(\sigma \upharpoonright n) \mid n \in \omega\}$ is a K-net of $C(x)$, i.e., if U is an open set in X such that $C(x) \subset U$, then $F(\sigma \upharpoonright n) \subset U$ for some n. Here $C(x) = \bigcap_{n\in\omega} F(\sigma \upharpoonright n)$.

If X is a paracompact Σ-space, then $C(x)$ is compact for each $x \in X$.

Lemma 1.3. *(Lemma 6 in [4]). If X is a metrizable space, then for each $n \in \omega$, there are locally finite open covers $\mathcal{V}_n$ and $\mathcal{B}_n$ of X satisfying the following conditions:*

(1) $\mathcal{V}_n = \{V(\sigma) \mid \sigma \in \Omega^n\}$, $\mathcal{B}_n = \{B(\sigma) \mid \sigma \in \Omega^n\}$,

(2) $\overline{B(\sigma)} \subset V(\sigma)$,

(3) $V(\sigma) = \bigcup_{\alpha\in\Omega} V(\sigma \vee \alpha)$, $B(\sigma) = \bigcup_{\alpha\in\Omega} B(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(4) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{V(\sigma \upharpoonright n) \mid n \in \omega\}$ is a local base of x in X and $\{B(\sigma \upharpoonright n) \mid n \in \omega\}$ is a local base of x in X.

We let $|A|$ denote the cardinality of a set A. For collections $\mathcal{G}$ and $\mathcal{H}$ of subsets of X, we let $\mathcal{G} \prec \mathcal{H}$ denote the condition that $\mathcal{G}$ is a refinement or a partial refinement of $\mathcal{H}$.

If a collection $\{F_\xi \mid \xi \in \Xi\}$ of subsets of X is locally finite (resp. discrete), then the collection $\{\overline{F_\xi} \mid \xi \in \Xi\}$ of closed subsets of X is locally finite (resp. discrete). Therefore in the definition of *expandable, discretely expandable, σ-expandable, etc.*, "$\{F_\xi \mid \xi \in \Xi\}$ *of subsets of* X" can be replaced by "$\{F_\xi \mid \xi \in \Xi\}$ *of closed subsets of* X".

2. σ-EXPANDABILITY

A space X is called *σ-expandable* (resp. *discretely σ-expandable*) [13] if for every locally finite (resp. discrete) collection $\{F_\xi \mid \xi \in \Xi\}$ of subsets in X, there exists a sequence $\{\mathcal{G}_n = \{G_{\xi,n} \mid \xi \in \Xi\} \mid n \in \omega\}$ of collections of open subsets in X satisfying the following:

(i) $F_\xi \subset \bigcup_{n\in\omega} G_{\xi,n}$ for each ξ.
(ii) $\mathcal{G}_n$ is locally finite in X.

Let us call $\{\mathcal{G}_n \mid n \in \omega\}$ a σ-expansion of $\mathcal{F}$.

Theorem 2.1. *Let X be a normal P-space and Y a paracompact Σ-space. If X is σ-expandable, then so is $X \times Y$.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a spectral Σ-net of Y, i.e., for some set Ω, $\mathcal{F}_n = \{F(\sigma) \mid \sigma \in \Omega^n\}$ is a locally finite closed cover of Y for each $n \in \omega$ satisfying the conditions in Lemma 1.2.

Since Y is paracompact, there is a locally finite open cover $\mathcal{V}_n = \{V(\sigma) \mid \sigma \in \Omega^n\}$ of Y such that $F(\sigma) \subset V(\sigma)$ for each $\sigma \in \Omega^n$.

Let $\mathcal{A} = \{A_\xi \mid \xi \in \Xi\}$ be a locally finite collection of closed subsets in $X \times Y$. For each $\sigma \in \Omega^{<\omega}$, put

$$U(\sigma) = \bigcup\{U \mid U \text{ is open in } X,\ (U \times F(\sigma)) \cap A_\xi \neq \emptyset \text{ for only finitely many } \xi \in \Xi\}.$$

Then
(1) $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\alpha \in \Omega$,
(2) for $\sigma \in \Omega^\omega$, if $\{F(\sigma \restriction n) \mid n \in \omega\}$ is a K-net of $C(y)$ for some $y \in Y$, then $\bigcup_{n\in\omega} U(\sigma \restriction n) = X$.

(1) follows from $F(\sigma \vee \alpha) \subset F(\sigma)$.

Proof of (2). Let $x \in X$. Let $y \in Y$ be in (2). Since $C(y)$ is compact, there are an open set U in X and an open set V of Y such that $\{x\} \times C(y) \subset U \times V$, $(U \times V) \cap A_\xi \neq \emptyset$ for only finitely many $\xi \in \Xi$. There is an n such that $C(y) \subset F(\sigma \restriction n) \subset V$. Then $(U \times F(\sigma \restriction n)) \cap A_\xi \neq \emptyset$ for only finitely many $\xi \in \Xi$. By the definition of $U(\sigma \restriction n)$, $U \subset U(\sigma \restriction n)$. Thus $x \in U(\sigma \restriction n)$. □

Since X is a normal P-space, there are collections $\{K(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of closed subsets and $\{W(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of open subsets in X such that
(3) $K(\sigma) \subset W(\sigma), \overline{W(\sigma)} \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,
(4) for each $\sigma \in \Omega^{\omega}$, if $\bigcup_{n\in\omega} U(\sigma \restriction n) = X$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = X$.

Put $M_n = \bigcup_{\sigma\in\Omega^n}(K(\sigma) \times F(\sigma))$. Then
(5) $\bigcup_{n\in\omega} M_n = X \times Y$.

Proof. Let $(x, y) \in X \times Y$ and let $\sigma \in \Omega^{\omega}$ with $\{F(\sigma \restriction n) \mid n \in \omega\}$ be a K-net of $C(y)$ for some $y \in Y$. Then, by (2) and (4), $\bigcup K(\sigma \restriction n) = X$. Thus $(x, y) \in K(\sigma \restriction n) \times F(\sigma \restriction n)$ for some n. □

For each $\sigma \in \Omega^{<\omega}$, let $\pi_\sigma : X \times F(\sigma) \to X$ be the projection. Put $L_{\xi,\sigma} = \pi_\sigma(A_\xi \cap (U(\sigma) \times F(\sigma)))$ and $L(\xi, \sigma) = \overline{L_{\xi,\sigma}} \cap \overline{W(\sigma)}$ for each $\xi \in \Xi$. Then
(6) $\mathcal{L}_\sigma = \{L(\xi, \sigma) \mid \xi \in \Xi\}$ is a locally finite collection of closed subsets in X.

Proof. It is sufficient to show that $\mathcal{L}_\sigma$ is locally finite in $\overline{W(\sigma)}$. Let $x \in \overline{W(\sigma)}$. Then $x \in U(\sigma)$ and therefore there is an open set U of X such that $x \in U \subset U(\sigma)$ and $(U \times F(\sigma)) \cap A_\xi \neq \emptyset$ for only finitely many $\xi \in \Xi$. Let $\xi \in \{\xi_i \mid i < k\} \iff (U \times F(\sigma)) \cap A_\xi \neq \emptyset$. If $L_{\xi,\sigma} \cap U \neq \emptyset$, let $x' \in L_{\xi,\sigma} \cap U$. Then there is an element $y' \in F(\sigma)$ such that $(x', y') \in (U \times F(\sigma)) \cap A_\xi$. Therefore $(U \times F(\sigma)) \cap A_\xi \neq \emptyset$ and so $\xi \in \{\xi_i \mid i < k\}$. Thus, if $\xi \notin \{\xi_i \mid i < k\}$, then $L_{\xi,\sigma} \cap U = \emptyset$, and so $\overline{L_{\xi,\sigma}} \cap U = \emptyset$. Hence $L(\xi, \sigma) \cap U = \emptyset$ if $\xi \notin \{\xi_i \mid i < k\}$. □

Since X is σ-expandable, there are locally finite collections $\mathcal{G}_{\sigma,m} = \{G_{\xi,\sigma,m} \mid \xi \in \Xi\}$ of open sets in X such that $L(\xi, \sigma) \subset \bigcup_{m\in\omega} G_{\xi,\sigma,m}$ for each $\xi \in \Xi$.

Let $H_{\xi,\sigma,m} = (G_{\xi,\sigma,m} \cap W(\sigma)) \times V(\sigma)$ and $H(\xi, n, m) = \bigcup_{\sigma\in\Omega^n} H_{\xi,\sigma,m}$. Put $\mathcal{H}(n, m) = \{H(\xi, n, m) \mid \xi \in \Xi\}$. Then
(7) $\mathcal{H}(n, m)$ is a locally finite collection of open subsets in $X \times Y$ for each $n, m \in \omega$.
(8) $A_\xi \subset \bigcup_{n,m\in\omega} H(\xi, n, m)$ for each $\xi \in \Xi$.

Proof of (7). Let $(x, y) \in X \times Y$. Since $\{V(\sigma) \mid \sigma \in \Omega^n\}$ is locally finite in Y, there is an open neighborhood O_2 of y in Y such that $O_2 \cap V(\sigma_i) \neq \emptyset$ for $\sigma_i \in \Omega^n, i < k$ and $O_2 \cap V(\sigma) = \emptyset$ for each $\sigma \in \Omega^n \setminus \{\sigma_i \mid i < k\}$. Since $\mathcal{G}_{\sigma_i,m}$ is a locally finite collection in

X for each $i < k$, there is an open neighborhood $O_{1,i}$ of x in X such that $O_{1,i} \cap G_{\xi,\sigma_i,m} \neq \emptyset$ for at most finitely many $\xi \in \Xi$. Then $O = \bigcap_{i<k} O_{1,i} \times O_2$ is an open neighborhood of (x, y) in $X \times Y$ such that $O \cap H(\xi, n, m) \neq \emptyset$ for finitely many $\xi \in \Xi$. □

Proof of (8). Let $(x, y) \in A_\xi$. Then $(x, y) \in K(\sigma) \times F(\sigma)$ for some $\sigma \in \Omega^{<\omega}$. Since $x \in L_{\xi,\sigma} \cap W(\sigma)$, $x \in L(\xi, \sigma)$ and so $x \in G_{\xi,\sigma,m}$ for some $m \in \omega$. Thus, $(x, y) \in H_{\xi,\sigma,m} \subset H(\xi, n, m)$ if $\sigma \in \Omega^n$. □

Since $|\omega \times \omega| = \omega$, $\{\mathcal{H}(n, m) \mid n, m \in \omega\}$ is a σ-expansion of $\mathcal{A}$. □

Theorem 2.2. *Let X be a normal P-space and Y a metrizable space. If X is discretely σ-expandable, then so is $X \times Y$.*

Proof. Let $\mathcal{V}_n$ and $\mathcal{B}_n$ be locally finite open covers of Y satisfying the conditions in Lemma 1.3.

Let $\mathcal{A} = \{A_\xi \mid \xi \in \Xi\}$ be a discrete collection of closed subsets in $X \times Y$. For each $\sigma \in \Omega^{<\omega}$, put

$$U(\sigma) = \bigcup\{U \mid U \text{ is open in } X, \ (U \times V(\sigma)) \cap A_\xi \neq \emptyset \text{ for at most only } \xi \in \Xi\}.$$

Then

(1) $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\alpha \in \Omega$,

(2) for $\sigma \in \Omega^\omega$, if $\{V(\sigma \upharpoonright n) \mid n \in \omega\}$ is a local base of $y \in Y$, then $\bigcup_{n\in\omega} U(\sigma \upharpoonright n) = X$.

Since X is a normal P-space, there are collections $\{K(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of closed subsets and $\{W(\sigma) \mid \sigma \in \Omega^{<\omega}\}$ of open subsets in X such that

(3) $K(\sigma) \subset W(\sigma), \overline{W(\sigma)} \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,

(4) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} U(\sigma \upharpoonright n) = X$, then $\bigcup_{n\in\omega} K(\sigma \upharpoonright n) = X$.

Put $Z_n = \bigcup_{\sigma\in\Omega^n}(K(\sigma) \times \overline{B(\sigma)})$. Then

(5) $\{Z_n \mid n \in \omega\}$ is a closed cover of $X \times Y$.

Proof. Since $\{\overline{B(\sigma)} \mid n \in \omega\}$ is locally finite, Z_n is closed in $X \times Y$. To show that $\bigcup_{n\in\omega} Z_n = X \times Y$, let $(x, y) \in X \times Y$ and let $\sigma \in \Omega^\omega$ with $\{V(\sigma \upharpoonright n) \mid n \in \omega\}$ be a local base of y in Y and $\{B(\sigma \upharpoonright n) \mid n \in \omega\}$ be a local base of y in Y. Then, by (2) and (4), $\bigcup K(\sigma \upharpoonright n) = X$. Thus $(x, y) \in K(\sigma \upharpoonright n) \times B(\sigma \upharpoonright n)$ for some n. □

For each $\sigma \in \Omega^{<\omega}$, let $\pi_\sigma : X \times V(\sigma) \to X$ be the projection. Put $L_{\xi,\sigma} = \pi_\sigma(A_\xi \cap (U(\sigma) \times V(\sigma)))$ and $L(\xi,\sigma) = \overline{L_{\xi,\sigma}} \cap \overline{W(\sigma)}$ for each $\xi \in \Xi$. Then

(6) $\mathcal{L}_\sigma = \{L(\xi,\sigma) \mid \xi \in \Xi\}$ is a discrete collection of closed subsets in X.

Proof. It is sufficient to show that $\mathcal{L}_\sigma$ is discrete in $\overline{W(\sigma)}$. Let $x \in \overline{W(\sigma)}$. Then $x \in U(\sigma)$ and therefore there is an open set U of X and $(U \times V(\sigma)) \cap A_\xi \neq \emptyset$ for at most only $\xi \in \Xi$. Let $\xi = \xi_0 \iff (U \times V(\sigma)) \cap A_\xi \neq \emptyset$. If $L_{\xi,\sigma} \cap U \neq \emptyset$, let $x' \in L_{\xi,\sigma} \cap U$. Then there is an element $y \in V(\sigma)$ such that $(x', y) \in (U \times V(\sigma)) \cap A_\xi$. Therefore $(U \times V(\sigma)) \cap A_\xi \neq \emptyset$ and so $\xi = \xi_0$. Thus, if $\xi \neq \xi_0$, then $L_{\xi,\sigma} \cap U = \emptyset$, and so $\overline{L_{\xi,\sigma}} \cap U = \emptyset$. Hence $L(\xi,\sigma) \cap U = \emptyset$ if $\xi \neq \xi_0$. □

After that, the proof is quite similar to that of Theorem 2.1. □

3. θ-EXPANDABILITY

A space X is called *θ-expandable* (resp. *discretely θ-expandable*) [7] if for every locally finite collection (resp. discrete collection) $\mathcal{F} = \{F_\xi \mid \xi \in \Xi\}$ of subsets in X, there exists a sequence $\{\mathcal{G}_n = \{G_{\xi,n} \mid \xi \in \Xi\} \mid n \in \omega\}$ of collections of open subsets in X satisfying the following:

(i) $F_\xi \subset G_{\xi,n}$ for each ξ and each n,

(ii) for every point x of X, there is an n_x for which x is contained in at most finitely many members of $\mathcal{G}_{n_x}$ (i.e., $\mathcal{G}_{n_x}$ is point finite at x).

Let us call $\{\mathcal{G}_n \mid n \in \omega\}$ a θ-expansion of $\mathcal{F}$.

Theorem 3.1. *Let X be a normal P-space and Y a paracompact Σ-space. If X is θ-expandable, then so is $X \times Y$.*

Proof. Let us define $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{V}_n$, $\mathcal{A}$, $U(\sigma)$, $K(\sigma)$, $W(\sigma)$, M_n and $\mathcal{L}_\sigma$ as in the proof of Theorem 2.1. Then (1) $\sim$ (6) as in Theorem 2.1 hold.

Since X is θ-expandable, there is a sequence of collections $\{\mathcal{G}_{\sigma,m} = \{G_{\xi,\sigma,m} \mid \xi \in \Xi\} \mid m \in \omega\}$ of open subsets in X for each $\sigma \in \Omega^{<\omega}$ such that

(i) $L(\xi,\sigma) \subset G_{\xi,\sigma,m}$,

(ii) for every point x of X, there is an m_x for which x is contained in at most finitely many members of $\mathcal{G}_{\sigma,m_x}$ (i.e., $\mathcal{G}_{\sigma,m_x}$ is point finite at x).

Put $G'_{\xi,\sigma,m} = G_{\xi,\sigma,m} \cap W(\sigma)$ and $\mathcal{G}'_{\sigma,m} = \{G'_{\xi,\sigma,m} \mid \xi \in \Xi\}$.

Let Ω^n be a well ordered set with $\prec$. Put
$\Omega(n,k) = \{(\sigma_0, \sigma_1, ..., \sigma_{k-1}) \mid \sigma_i \in \Omega^n, i < k; \sigma_0 \prec \sigma_1 \prec ... \prec \sigma_{k-1}\}$. For $\overline{m} = (m_0, m_1, ..., m_{k-1}) \in \omega^k$ and $\overline{\sigma} = (\sigma_0, \sigma_1, ..., \sigma_{k-1}) \in \Omega(n,k)$, define $H(\xi, n, k, \overline{\sigma}, \overline{m}) = \bigcup_{i<k}(G'_{\xi,\sigma_i,m_i} \times (V(\sigma_i) \smallsetminus E(\overline{\sigma}))$ where $E(\overline{\sigma}) = \bigcup\{F(\sigma') \mid \sigma' \in \Omega^n \smallsetminus \{\sigma_i \mid i < k\}\}$.

Put $\Lambda = \bigcup_{n,k\in\omega}(\{n\} \times \{k\} \times \omega^k)$. Then $|\Lambda| = \omega$.

For each $\lambda = (n, k, \overline{m}) \in \Lambda$, put $H(\xi, \lambda; \overline{\sigma}) = H(\xi, n, k, \overline{\sigma}, \overline{m})$ and $H_{\xi,\lambda} = \bigcup\{H(\xi, \lambda; \overline{\sigma}) \mid \overline{\sigma} \in \Omega(n,k)\} \cup (X \times Y \smallsetminus M_n)$. Put $\mathcal{H}_\lambda = \{H_{\xi,\lambda} \mid \xi \in \Xi\}$.

Then $H_{\xi,\lambda}$ is an open set in $X \times Y$ and

(7) $A_\xi \subset H_{\xi,\lambda}$,

(8) for each $(x, y) \in X \times Y$, there is a $\lambda \in \Lambda$ such that $\mathcal{H}_\lambda$ is point finite at (x, y).

Proof of (7). Let $\overline{m} = (m_0, m_1, ..., m_{k-1})$. It is obvious that $A_\xi \cap (X \times Y \smallsetminus M_n) \subset H_{\xi,\lambda}$. Let $(x, y) \in A_\xi \cap M_n$. Then $(x, y) \in K(\sigma) \times F(\sigma)$ for some $\sigma \in \Omega^n$. Since $\{V(\sigma) \mid \sigma \in \Omega^n\}$ is a locally finite open cover of Y, there is a finite subset $\{\sigma_i \mid i < k\}$ of Ω^n such that $y \in V(\sigma) \iff \sigma \in \{\sigma_i \mid i < k\}$. Since $F(\sigma) \subset V(\sigma)$ for each σ, $y \notin F(\sigma)$ for each $\sigma \in \Omega^n \smallsetminus \{\sigma_i \mid i < k\}$. Therefore we may assume that $(x, y) \in K(\sigma_0) \times F(\sigma_0)$ without loss of generality. Then $x \in W(\sigma_0) \cap \pi_{\sigma_0}(A_\xi \cap (U(\sigma_0) \times F(\sigma_0)) \subset L(\xi, \sigma_0) \subset G_{\xi,\sigma_0,m_0}$. Put $\overline{\sigma} = (\sigma_0, \sigma_1, ..., \sigma_{k-1})$. Then $(x, y) \in G'_{\xi,\sigma_0,m_0} \times (V(\sigma_0) \smallsetminus E(\overline{\sigma})) \subset H(\xi, \lambda; \overline{\sigma}) \subset H_{\xi,\lambda}$. □

Proof of (8). Let $(x, y) \in X \times Y$. By (5), $(x, y) \in M_n$ for some $n \in \omega$. Since $\{F(\sigma) \mid \sigma \in \Omega^n\}$ is a locally finite cover of Y, there is a finite subset $\{\sigma_i \mid i < k\}$ of Ω^n such that $y \in F(\sigma) \iff \sigma \in \{\sigma_i \mid i < k\}$. We may assume that $\sigma_0 \prec \sigma_1 \prec ..., \prec \sigma_{k-1}$. For each $i < k$, there is an $m_i \in \omega$ such that $\mathcal{G}_{\sigma_i,m_i}$ is point finite at x. Let $\lambda = (n, k, \overline{m})$, $\overline{m} = (m_0, m_1, ..., m_{k-1})$. Then $\mathcal{H}_\lambda$ is point finite at (x, y).

To show this, let $(x, y) \in H_{\xi,\lambda}$. Then $(x, y) \in H(\xi, \lambda; \overline{\sigma'})$ for some $\overline{\sigma'} = (\sigma'_0, \sigma'_1, ..., \sigma'_{k-1}) \in \Omega(n,k)$. Here we have $\{\sigma_i \mid i < k\} = \{\sigma'_i \mid i < k\}$. To see this, suppose $\{\sigma_i \mid i < k\} \neq \{\sigma'_i \mid i < k\}$.

Then there is a σ_j with $\sigma_j \notin \{\sigma_i' \mid i < k\}$. Since $y \notin E(\overline{\sigma'})$, $y \notin F(\sigma_j)$. This is a contradiction. Since $\sigma_0 \prec \sigma_1 \prec ... \prec \sigma_{k-1}$ and $\sigma_0' \prec \sigma_1' \prec ... \prec \sigma_{k-1}'$, $\sigma_i = \sigma_i'$ for each $i < k$. Thus, $\overline{\sigma} = \overline{\sigma'}$.

Put $\Xi_i = \{\xi \in \Xi \mid (x, y) \in G_{\xi,\sigma_i,m_i} \times (V(\sigma_i) \setminus E(\overline{\sigma}))\}$ for each i. Then Ξ_i is a finite subset and so $\Xi' = \cup_{i<k}\Xi_i$ is a finite subset of Ξ. If $(x, y) \in H_{\xi,\lambda}$, then $(x, y) \in H(\xi, \lambda; \overline{\sigma})$. Thus $\xi \in \Xi'$. □

Since $|\Lambda| = \omega$, $\{\mathcal{H}_\lambda \mid \lambda \in \Lambda\}$ is a θ-expansion of $\mathcal{A}$. □

Theorem 3.2. *Let X be a normal P-space and Y a metrizable-space. If X is discretely θ-expandable, then so is $X \times Y$.*

Proof. Let us define $\mathcal{V}_n$, $\mathcal{B}_n$, $\mathcal{A}$, $U(\sigma)$, $K(\sigma)$, $W(\sigma)$, Z_n and $\mathcal{L}_\sigma$ as in the proof of Theorem 2.2. Then (1) $\sim$ (6) as in Theorem 2.2 hold.

After that, the proof is quite similar to that of Theorem 3.1. □

4. Subexpandability and discretely subexpandability

A space X is called *discretely subexpandable* [7] if for every discrete collection $\{F_\xi \mid \xi \in \Xi\}$ of subsets of X, there exists a sequence $\{\mathcal{G}_n = \{G_{\xi,n} \mid \xi \in \Xi\} \mid n \in \omega\}$ of collections of open subsets of X satisfying the following:
(i) $F_\xi \subset G_{\xi,n}$ for each ξ and each n,
(ii) for every point x of X, there is an $n \in \omega$ for which x is contained in at most one member of $\mathcal{G}_n$.

The definition of discretely subexpandable coincides with the notion of *collectionwise subnormal* in [2].

A space X is *countably subparacompact* if every countable open covering of X has a σ-discrete closed refinement.

A space X is called *subexpandable* [7] if X is discretely subexpandable and countably subparacompact.

Theorem 4.1. *Let X be a normal P-space and Y a metrizable space. If X is discretely subexpandable, then so is $X \times Y$.*

Proof. Let $\mathcal{V}_n$ and $\mathcal{B}_n$ be locally finite open covers of Y satisfying the conditions in Lemma 1.3.

Let $\mathcal{A} = \{A_\xi \mid \xi \in \Xi\}$ be a discrete collection of closed subsets in $X \times Y$. For each $\sigma \in \Omega^{<\omega}$, put

$$U(\sigma) = \bigcup\{U \mid U \text{ is open in } X,\ (U \times V(\sigma)) \cap A_\xi \neq \emptyset \text{ for at most only } \xi \in \Xi\}.$$

Let us define $K(\sigma)$ and $W(\sigma)$ as in the proof of Theorem 2.2. Then

(4-1-1) $\bigcup_{n\in\omega}\bigcup_{\sigma\in\Omega^n}(K(\sigma)\times B(\sigma)) = X\times Y$. (cf. Proof of (5) in Theorem 2.2)

Put $\mathcal{O}_n = \{O(y) \mid O(y)$ is an open neighborhood of y in Y, $O(y)\cap V(\sigma)\neq\emptyset$ for only finitely many $\sigma\in\Omega^n\}$. Then, since $\mathcal{V}_n$ is a locally finite open cover of Y, $\mathcal{O}_n$ is an open cover of Y. Since Y is metrizable, there is a σ-discrete open cover $\mathcal{D}_n = \bigcup_{l\in\omega}\mathcal{D}_{n,l}$ of Y such that $\mathcal{D}_n \prec \mathcal{O}_n$, each $\mathcal{D}_{n,l}$ is discrete in Y. And there is an open cover $\mathcal{D}^{'}_n$ of Y which is a shrinking of $\mathcal{D}_n$, i.e., $\mathcal{D}^{'}_n = \{D^{'} \mid D\in\mathcal{D}_n\}$ with $\overline{D^{'}}\subset D$ for each $D\in\mathcal{D}_n$. For each $D\in\mathcal{D}_n$, $D\cap V(\sigma)\neq\emptyset$ for only finitely many $\sigma\in\Omega^n$. Put $\mathcal{E}_D = \{D\cap V(\sigma) \mid \sigma\in\Omega^n, D\cap V(\sigma)\neq\emptyset\}$. Since $\mathcal{E}_D$ is a finite set, represent $\mathcal{E}_D = \{D_i \mid i<k\}$ for some $k\in\omega$. By setting $D_i=\emptyset$ for each $i\geq k$, we can represent $\mathcal{E}_D = \{D_i \mid i\in\omega\}$. Put $\mathcal{E}_{n,l,i} = \{D_i \mid D\in\mathcal{D}_{n,l}\}$ for each $n,l,i\in\omega$. Put $\mathcal{E} = \bigcup_{n,l,i\in\omega}\mathcal{E}_{n,l,i}$. Since $|\omega\times\omega\times\omega| = \omega$, we can represent $\mathcal{E} = \bigcup_{m\in\omega}\mathcal{E}_m$.

For each $E\in\mathcal{E}$, $E = D\cap V(\sigma)$ for some $D\in\mathcal{D}_n$ (for some $n\in\omega$) and some $\sigma\in\Omega^n$. Such D and σ are uniquely decided for E. Define $K_E = K(\sigma), U_E = U(\sigma), B_E = D^{'}\cap B(\sigma)$ and $W_E = W(\sigma)$.

Put $T_m = \bigcup_{E\in\mathcal{E}_m}(K_E\times\overline{B_E})$ for each $m\in\omega$. Then

(4-1-2) T_m is a closed set in $X\times Y$ for each $m\in\omega$.

Proof. Since $B_E\subset E$ for each $E\in\mathcal{E}_m$ and $\mathcal{E}_m$ is discrete in Y, $\{B_E \mid E\in\mathcal{E}_m\}$ is discrete in Y. Therefore $\{K_E\times\overline{B_E} \mid E\in\mathcal{E}_m\}$ is a discrete collection of closed subsets in $X\times Y$. □

(4-1-3) $\bigcup_{m\in\omega} T_m = X\times Y$.

Proof. Let $(x,y)\in X\times Y$. Then $(x,y)\in K(\sigma)\times B(\sigma)$ for some $\sigma\in\Omega^n$. Since $\mathcal{D}^{'}_n$ is a cover of Y, there is a $D^{'}\in\mathcal{D}^{'}_n$ such that $y\in D^{'}$. Then $y\in D\cap V(\sigma)$. Let $D\cap V(\sigma) = E$. Then $K_E = K(\sigma)$, $B_E = D^{'}\cap B(\sigma)$. Therefore $(x,y)\in K_E\times B_E$. □

For each $E\in\mathcal{E}$, let $\pi_E : X\times E\to X$ be the projection. Put $F_{\xi,E} = \pi_E((U_E\times E)\cap A_\xi)$, $F(\xi,E) = \overline{F_{\xi,E}}\cap\overline{W_E}$ and $\mathcal{F}_E = \{F(\xi,E) \mid \xi\in\Xi\}$. Then

(4-1-4) $\mathcal{F}_E$ is a discrete collection of closed subsets in X.

Proof. It is sufficient to show that $\mathcal{F}_E$ is discrete in $\overline{W_E}$. Put $E = D \cap V(\sigma)$, $D \in \mathcal{D}_n$, $\sigma \in \Omega^n$. Let $x \in \overline{W_E}$. Then $x \in U_E = U(\sigma)$ and therefore there is an open set U of X with $x \in U \subset U(\sigma)$ and $(U \times V(\sigma)) \cap A_\xi = \emptyset$ for each $\xi \in \Xi$, $\xi \neq \xi_0$. If $F_{\xi,E} \cap U \neq \emptyset$, let $x' \in F_{\xi,E} \cap U$. Then there is an element $y' \in Y$ such that $(x', y') \in (U \times E) \cap A_\xi$. Therefore $(U \times V(\sigma)) \cap A_\xi \neq \emptyset$ and so $\xi = \xi_0$. Thus, if $\xi \neq \xi_0$, then $F_{\xi,E} \cap U = \emptyset$, and so $\overline{F_{\xi,E}} \cap U = \emptyset$. Hence $F(\xi, E) \cap U = \emptyset$ if $\xi \neq \xi_0$. □

Since X is discretely subexpandable, there is a sequence $\{\mathcal{G}_{E,i} = \{G(\xi, E, i) \mid \xi \in \Xi\} \mid i \in \omega\}$ of collections of open subsets in X such that
$(i)_E$. $F(\xi, E) \subset G(\xi, E, i)$ for each $i \in \omega$,
$(ii)_E$. for each $x \in X$, there is an $i_x \in \omega$ such that $\operatorname{ord}(x, \mathcal{G}_{E,i_x}) \leq 1$.

Let $G(\xi, E, i)' = G(\xi, E, i) \cap W_E$ and

$H_{\xi,m,i} = (\bigcup_{E \in \mathcal{E}_m} (G(\xi, E, i)' \times E)) \cup (X \times Y \setminus T_m)$.

Put $\mathcal{H}_{m,i} = \{H_{\xi,m,i} \mid \xi \in \Xi\}$ for each $m, i \in \omega$. Then

(i) $A_\xi \subset H_{\xi,m,i}$ for each ξ,
(ii) for each $(x, y) \in X \times Y$, there are an m and an $i \in \omega$ such that $\operatorname{ord}((x, y), \mathcal{H}_{m,i}) \leq 1$.

Proof of (i). Let $(x, y) \in A_\xi$. If $(x, y) \notin T_m$, it is obvious that $(x, y) \in H_{\xi,m,i}$. If $(x, y) \in T_m$, then $(x, y) \in K_E \times \overline{B_E}$ for some $E \in \mathcal{E}_m$. Since $K_E \subset W_E \subset U_E$, $\overline{B_E} \subset E$, $(x, y) \in (U_E \times E) \cap A_\xi$. Thus $x \in F(\xi, E)$ and so $(x, y) \in G(\xi, E, i)' \times E \subset H_{\xi,m,i}$. □

Proof of (ii). Let $(x, y) \in X \times Y$. Then $(x, y) \in K_E \times \overline{B_E}$ for some $E \in \mathcal{E}_m$. Since $\mathcal{E}_m$ is discrete in Y, $y \notin E'$ for each $E' \in \mathcal{E}_m$, $E' \neq E$. Choose $i \in \omega$ such that $\operatorname{ord}(x, \mathcal{G}_{E,i}) \leq 1$. Then $\operatorname{ord}((x, y), \mathcal{H}_{m,i}) \leq 1$. To show this, choose $\xi \in \Xi$ with $x \notin G(\xi', E, i)$ if $\xi' \neq \xi$ and let $(x, y) \in H_{\xi',m,i}$. Then $(x, y) \in G(\xi', E', i) \times E'$ for some $E' \in \mathcal{E}_m$. Then $E' = E$. Since $\operatorname{ord}(x, \mathcal{G}_{E,i}) \leq 1$, $\xi' = \xi$. □

Theorem 4.2. *Let X be a normal P-space and Y a metrizable space. If X is subexpandable, then so is $X \times Y$.*

Proof. It is well known that $X \times Y$ is normal and countably paracompact and therefore countably subparacompact. This theorem follows from Theorem 4.1. □

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Department of Mathematics, Faculty of Science, Shizuoka University, Ohya, Shizuoka 422-8529

E-mail address: smktiba@ipc.shizuoka.ac.jp