TOPOLOGY PROCEEDINGS

Volume 37, 2011

Pages 145–153

http://topology.auburn.edu/tp/

Perfect images of submetric spaces

by

Huaipeng Chen

Electronically published on May 20, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 37 (2011)
Pages 145-153
E-Published on May 20, 2010

PERFECT IMAGES OF SUBMETRIC SPACES

HUAIPENG CHEN

ABSTRACT. The following theorem is obtained:

Theorem: Let (X, τ) be a paracompact σ-space and $f : (X, \tau) \to (Y, \tau_Y)$ be a perfect mapping. Then both (X, τ) and (Y, τ_Y) can be submetrized by metric ρ and d respectively such that $f : (X, \tau_\rho) \to (Y, \tau_d)$ is a perfect mapping.

1. INTRODUCTION

All spaces are regular T_1-spaces, all mappings are onto and continuous and letter N denotes the set of positive integers throughout this paper.

Recall that a space (X, τ) *submetrizable* [4] if there is a metric ρ on X with $\rho \subset \tau$. Note that such a space (X, τ) is *contractible onto a metric space* in the sense of Martin [5] since a space (X, τ) is *contractible onto a metric space* provided there exists a one to one and continuous map from X onto a metric space. To denote the induced topology on X by a metric ρ we shall use symbol $\tau\rho$.

Martin [5] gave some important theorems on submetric spaces and raised questions on perfect images of submetric spaces since it is well known that the perfect image of a metric space is metrizable.

Recall a map $f : X \to Y$ *perfect* if $f(C)$ is closed for each closed set C of X and $f^{-1}(y)$ is compact for each $y \in Y$.

2010 *Mathematics Subject Classification.* 54C10, 54E35.

Key words and phrases. Closed maps, perfect maps, metric spaces.

* Project supported by The National Natural Science Foundation of China (No.10471084).

Popov in [7] has shown that the perfect image of a hereditarily paracompact submetric space need not to be submetrizable (to see Note 2 in this paper). So we would like to ask the following question.

Question. Let (X,τ) be a paracompact σ-space and $f:(X,\tau)\to(Y,\tau_Y)$ be a perfect mapping. Then can (X,τ) and (Y,τ_Y) be submetrized by metric ρ and d respectively such that $f:(X,\tau_\rho)\to(Y,\tau_d)$ is a perfect mapping?

A *network* for a space X is a collection $\mathcal{F}$ of subset of X such that whenever $x\in U$ with U open, there exists $F\in\mathcal{F}$ with $x\in F\subset U$ [1]. A space X is a σ-space if X has a σ-discrete (equivalently, σ-locally finite) network [6].

In this paper, we give a positive answer to the above Question and raise a question.

2. Main results

Let (X,τ) be a topological space and $f:(X,\tau)\to(Y,\tau_Y)$ be a continuous mapping. We use Theorem 1.4.13 in [3] as the following Lemma 2.1.

Lemma 2.1. *A mapping $f:(X,\tau)\to(Y,\tau_Y)$ is closed if and only if for every point $y\in Y$ and every open set $U\subset X$ which contains $f^{-1}(y)$, there exists in Y a neighborhood V of the point y such that $f^{-1}(V)\subset U$.*

Take an $O\in\tau$ and let

$$S(O)=\cup\{f^{-1}(y):f^{-1}(y)\subset O\}\ \ \textit{and}\ \ \mathcal{E}=\{S(O):O\in\tau\}.$$

Symbols $S(O)$ and $\mathcal{E}$ are used always with the same meaning throughout this paper. The following Lemma 2.2 is well-known as characterization which are used always in this paper.

Lemma 2.2. *Let (X,τ) be a topological space and $f:(X,\tau)\to(Y,\tau_Y)$ be a continuous mapping. Then the following are equivalent:*
1) f is a closed mapping,
2) $\mathcal{E}\subset\tau$ and $\mathcal{U}=\{f(S(O)):O\in\tau\}$ and
3) If $O\in\tau$, then $S(O)\in\tau$ and $f(S(O))\in\mathcal{U}$.

We introduce the following definition for convenience.

Definition 2.3. Let both $\mathcal{V}$ and $\mathcal{O}$ be collections of open sets. Call $\mathcal{V}$ a quasi-base refinement of $\mathcal{O}$ if for each $O\in\mathcal{O}$ and each $x\in O$ there is an $V\in\mathcal{V}$ with $x\in V\subset O$.

Lemma 2.4. *Let X be a paracompact space and $\mathcal{O}$ be a locally finite open cover of X. Then $\mathcal{O}$ has a σ-discrete quasi-base refinement.*

Proof. Let $\mathcal{O} = \{O_\alpha : \alpha \in \Lambda\}$. Then, for each $x \in X$, there exists an open neighborhood U_x of x such that U_x meets only finitely many O_α's in $\mathcal{O}$. Let $\mathcal{U}' = \{U_x : x \in X\}$. Then $\mathcal{U}'$ is an open cover of X. Let $\cup_n \mathcal{U}_n$ be a σ-discrete refinement of $\mathcal{U}'$. We denote by $\mathcal{U}_n = \{U_{n\alpha} : \alpha \in \Lambda(n)\}$ and take any $U_{n\alpha} \in \mathcal{U}_n$. Let $\mathcal{O}_{n\alpha} = \{O \in \mathcal{O} : O \cap U_{n\alpha} \neq \emptyset\}$. Then $\mathcal{O}_{n\alpha}$ is finite. So $\mathcal{O}_{n\alpha} = \{O_{n\alpha i} : i < i(\alpha)\}$. Let $V_{n\alpha i} = O_{n\alpha i} \cap U_{n\alpha}$ and $\mathcal{V}_{ni} = \{V_{n\alpha i} : \alpha \in \Lambda(n)\}$. Then $\mathcal{V}_{ni}$ is discrete since for each $V_{n\alpha i} \in \mathcal{V}_{ni}$, $V_{n\alpha i} \subset U_{n\alpha}$ and $\mathcal{U}_n = \{U_{n\alpha} : \alpha \in \Lambda(n)\}$ is discrete. Let $\mathcal{V} = \cup_{ni} \mathcal{V}_{ni}$. Then $\mathcal{V}$ is a σ-discrete quasi-base refinement of $\mathcal{O}$. □

In this section we shall use a specified base for metrizable spaces called a fine base.

Definition 2.5. Let (X, τ_ρ) be a metric space. Call a collection $\mathcal{B} = \cup_n \mathcal{B}_n$ of open sets in (X, τ_ρ) a fine base of metric ρ if:

1) each $\mathcal{B}_n$ is locally finite in (X, τ_ρ) and $\cup \mathcal{B}_n = X$,
2) $diam(B_{n\alpha}) < 1/n$ for each $B_{n\alpha} \in \mathcal{B}_n$, and
3) $\mathcal{B}_n$ is closed to finite intersections.

Recall a *set G a ρ-compact set* if and only if G is a compact set in (X, τ_ρ). Similarly, G has property $\rho - P$ if G has the property P in (X, τ_ρ).

Let (X, τ) be a paracompact submetric space, $f : (X, \tau) \to (Y, \tau_Y)$ be a perfect mapping and $\mathcal{B} = \cup_n \mathcal{B}_n$ be a fine-base of a metrizable topology ρ.

For each n and each $y \in Y$, let

$$\mathcal{B}_n = \{B_{n\beta} : \beta \in \Lambda(n)\} \text{ and } \mathcal{B}_{ny} = \{B_{n\beta} \in \mathcal{B}_n : f^{-1}(y) \cap B_{n\beta} \neq \emptyset\}.$$

Then $\mathcal{B}_{ny}$ is finite since $f^{-1}(y)$ is compact for every y and $B_{n\beta}$'s are open in (X, τ).

Recall a cover $\mathcal{B}' \subset \mathcal{B}_{ny}$ of $f^{-1}(y)$ an *irreducible* cover if any proper subfamily $\mathcal{B}'' \subset \mathcal{B}'$ does not cover $f^{-1}(y)$. Let $\mathbb{B}_{ny}$ be a family of all irreducible finite cover of $f^{-1}(y)$ from $\mathcal{B}_{ny}$. Since $\mathcal{B}_{ny}$ is finite, $\mathbb{B}_{ny}$ is also finite. We denote by $\mathbb{B}_{ny} = \{\mathcal{B}_{nyi} : i \leq n(y)\}$. For each $y \in Y$ and each $i \leq n(y)$, let $B_{nyi} = \cup \mathcal{B}_{nyi}$. Let

$$\mathcal{B}(n) = \{B_{nyi} : y \in Y \text{ and } i \leq n(y)\},$$

$$\mathcal{B}(n,k)=\{B_{nyi}\in\mathcal{B}(n):|\mathcal{B}_{nyi}|=k\},$$
$$S(B_{nyi})=\cup\{f^{-1}(y'):f^{-1}(y')\subset B_{nyi}\}$$
$$\text{and}\quad \mathcal{E}(n,k)=\{S(B_{nyi}):B_{nyi}\in\mathcal{B}(n,k)\}.$$

Then, for each $y\in Y$, there is the least number $k\in N$ with $f^{-1}(y)\subset S(B_{nyi})$ and $S(B_{nyi})\in\mathcal{E}(n,k)$. Then $S(B_{nyi})\in\tau$ by Lemma 2.2 since $B_{nyi}\in\tau$ and $f:(X,\tau)\to(Y,\tau_Y)$ is a closed mapping.

Proposition 2.6. *Let (X,τ) be a paracompact submetric space, $\mathcal{B}=\cup_n\mathcal{B}_n$ be a fine-base of the metrizable topology ρ and $f:(X,\tau)\to(Y,\tau_Y)$ be a perfect mapping. Then, for each n, there is a σ-discrete collection $\cup_k\mathcal{U}(n,k)\subset\mathcal{U}$ such that for each $f^{-1}(y)\subset X$, there exist a $k\in N$ and an $U_{kn\alpha}\in\mathcal{U}(n,k)$ with $f^{-1}(y)\subset f^{-1}(U_{kn\alpha})\subset B_{ny}$.*

Here $\mathcal{B}_{ny}=\{B_{n\beta}\in\mathcal{B}_n:f^{-1}(y)\cap B_{n\beta}\neq\emptyset\}$ and $B_{ny}=\cup\mathcal{B}_{ny}$.

Proof. We construct a σ-discrete collections $\cup_k f^{-1}(\mathcal{U}(n,k))$ of open sets in (X,τ) by induction on k for each n.

A. Take $\mathcal{E}(n,1)=\{S(B_{nyi}):B_{nyi}\in\mathcal{B}(n,1)\}$ with $\mathcal{B}_{nyi}=\{B_{n\alpha}\}$ and $f^{-1}(y)\subset B_{n\alpha}$ since $|\mathcal{B}_{nyi}|=1$. Then $\mathcal{E}(n,1)$ is locally finite since $\mathcal{B}_n$ is locally finite and $S(B_{nyi})\subset B_{nyi}=B_{n\alpha}\in\mathcal{B}_n$ for each $S(B_{nyi})\in\mathcal{E}(n,1)$. Then $\mathcal{E}(n,1)$ is a collection of open sets by Lemma 2.2 since $f:(X,\tau)\to(Y,\tau_Y)$ is a perfect mapping. Let $\mathcal{E}'(n,1)=\mathcal{E}(n,1)$ and $f(\mathcal{E}'(n,1))=\{f(S'(B_{nyi})):S'(B_{nyi})\in\mathcal{E}'(n,1)\}$. Then $f(\mathcal{E}'(n,1))$ is a locally finite collection of open sets by Lemma 2.2. Then $f(\mathcal{E}'(n,1))$ has a σ-discrete quasi-base refinement $\mathcal{U}(n,1)=\cup_m\mathcal{V}_{1m}$ by Lemma 2.4 since (Y,τ_Y) is paracompact. Let $E(n,1)=\cup\mathcal{E}'(n,1)$. Then $E(n,1)\in\tau$.

So, for each $f^{-1}(y)\subset E(n,1)$, there exists an $S'(B_{nyj})\in\mathcal{E}'(n,1)$ and an $U_{1m\alpha}\in\mathcal{U}(n,1)$ with $f^{-1}(y)\subset f^{-1}(U_{1m\alpha})\subset S(B_{nyj})\subset B_{ny}$ since $\mathcal{U}(n,1)$ is a quasi-base refinement of $f(\mathcal{E}'(n,1))$.

B. Assume we have had $E(n,l)\in\tau$ and $\mathcal{U}(n,l)$ for each $l<k$ such that:

1) $\mathcal{U}(n,l)$ is a σ-discrete collection of open sets in Y and

2) for each $f^{-1}(y)\subset E(n,l)$, there is an $U_{lm\alpha}\in\mathcal{U}(n,l)$ with $f^{-1}(y)\subset f^{-1}(U_{lm\alpha})$ and $f^{-1}(U_{lm\alpha})\subset S(B_{nyi})\subset B_{ny}$.

Take $\mathcal{E}(n,k)=\{S(B_{nyi}):B_{nyi}\in\mathcal{B}(n,k)\}$. Let $E_k=\cup_{l<k}E(n,l)$. Then we have $E_k\in\tau$ and $f(E_k)\in\mathcal{U}$ by Lemma 2.2.

For each $S(B_{nyi}) \in \mathcal{E}(n,k)$, let $S'(B_{nyi}) = S(B_{nyi}) - E_k$ and $\mathcal{E}'(n,k) = \{S'(B_{nyi}) : B_{nyi} \in \mathcal{B}(n,k)\}$.

Claim 2.7. *$\mathcal{E}'(n,k)$ is locally finite in X.*

Proof. Pick an $x \in X$. Case 1, $x \in E_k$. Then we have $E_k \in \tau$ and $E_k \cap (\cup\mathcal{E}'(n,k)) = \emptyset$.

Case 2, $x \in X - E_k$. Let $f(x) = y$. Then $f^{-1}(y) \cap E_k = \emptyset$ by the definition of E_k. Take a B_{nyi} from $\mathcal{B}(n,k)$. Let

$$O_y = B_{nyi} - \cup\{Cl(B_{n\beta}) : B_{n\beta} \in \mathcal{B}_n \text{ and } Cl(B_{n\beta}) \cap f^{-1}(y) = \emptyset\}.$$

Then $S(O_y) \in \tau$ by Lemma 2.2 and $f^{-1}(y) \subset S(O_y)$ since we have $f^{-1}(y) \subset O_y$.

To prove that $\mathcal{E}'(n,k)$ is locally finite in $X - E_k$, we prove that $S(O_y)$ meets only finitely many $S'(B_{nyi})$'s in $\mathcal{E}'(n,k)$ for $f^{-1}(y) \subset X - E_k$. To do it pick an $x' \in S'(B_{nyi}) \cap S(O_y)$ if $S'(B_{nyi}) \cap S(O_y) \neq \emptyset$. Let $y' = f(x')$. Then $f^{-1}(y') \cap E_k = \emptyset$ by the definition of $S'(B_{nyi})$.

(a). $x' \in S'(B_{nyi}) \cap S(O_y) \subset S(O_y) - E_k$. Then we have $f^{-1}(y') \subset S(O_y) - E_k$.

(b). $x' \in S'(B_{nyi}) \cap S(O_y) \subset S(B_{nyi}) - E_k$. Then we have $f^{-1}(y') \subset S'(B_{nyi}) \subset B_{nyi}$ and $B_{nyi} = B_{n\delta_1} \cup B_{n\delta_2} \cup ... \cup B_{n\delta_k}$ since $S(B_{nyi}) \in \mathcal{E}(n,k)$. Then $f^{-1}(y') \cap B_{n\delta_i} \neq \emptyset$ for each $i \leq k$. (In fact, $f^{-1}(y') \cap B_{n\delta_k} = \emptyset$ implies $f^{-1}(y') \subset B_{n\delta_1} \cup ... \cup B_{n\delta_{k-1}}$. Then $f^{-1}(y') \subset E_k$ by the definition of E_k, a contradiction to $f^{-1}(y') \cap E_k = \emptyset$.)

Suppose $Cl(B_{n\delta_i}) \cap f^{-1}(y) = \emptyset$ for some $i \leq k$. Then we have $Cl(B_{n\delta_i}) \cap O_y = \emptyset$ by the definition of O_y. Then $Cl(B_{n\delta_i}) \cap S(O_y) = \emptyset$. Note $f^{-1}(y') \subset S(O_y)$ by (a). Then $f^{-1}(y') \cap Cl(B_{n\delta_i}) = \emptyset$, a contradiction to $f^{-1}(y') \cap B_{n\delta_i} \neq \emptyset$ for each $i \leq k$. Let $\mathcal{B}_{nyi} = \{B_{n\delta_i} : i \leq k\}$. Then $\mathcal{B}_{nyi} \in \mathbb{B}_{ny}$ and $B_{nyi} = \cup\mathcal{B}_{nyi}$. So we have proved that, for each $f^{-1}(y) \subset X - E_k$, $S'(B_{nyi}) \cap S(O_y) \neq \emptyset$ implies $B_{nyi} \in \mathbb{B}_{ny}$ for each $S'(B_{nyi}) \in \mathcal{E}'(n,k)$. Then $\mathcal{E}'(n,k)$ is locally finite in X since $\mathbb{B}_{ny}$ is finite. □

(*Continued proof of Proposition 2.6*) Then, by Claim 2.7, $\mathcal{E}'(n,k)$ is locally finite in X. Then $f(\mathcal{E}'(n,k))$ is locally finite in Y since $f : (X,\tau) \to (Y,\tau_Y)$ is a perfect mapping. Let $Y_k = Y - f(E_k)$. Then Y_k is a closed subspace of Y since $f(E_k) \in \mathcal{U}$. Note that we have had $f(S(B_{nyi})) \cap Y_k = f(S(B_{nyi})) - f(E_k) = f(S'(B_{nyi}))$.

Then $f(\mathcal{E}'(n,k))$ is a locally finite collection of open sets in Y_k. Then $f(\mathcal{E}'(n,k))$ has a σ-discrete quasi-base refinement $\mathcal{V}'_k = \cup_m \mathcal{V}'_{km}$ in Y_k by Lemma 2.4 since Y_k is a closed subspace of paracompact space Y. Then $\mathcal{V}^-_k = \{Cl(V) : V \in \mathcal{V}'_k\}$ is a σ-discrete collection of closed sets in Y.

Note that X is collectionwise normal since every paracompact space is collectionwise normal by Theorem 5.1.18 in [3]. Then there is a σ-discrete collection $\mathcal{V}_k = \cup_m \mathcal{V}_{km}$ of open sets in Y such that, for each $Cl(V'_{km\alpha}) \in \mathcal{V}^-_k$, there is a $V_{km\alpha} \in \mathcal{V}_k$ with $Cl(V'_{km\alpha}) \subset V_{km\alpha}$ by Theorem 5.1.17 in [3]. For each $V'_{km\alpha} \in \mathcal{V}'_{km}$, take an open set $O_{km\alpha}$ of Y. Then $V'_{km\alpha} \subset Cl(V'_{km\alpha}) \subset V_{km\alpha}$. Moreover we can have $S(B_{nyj}) \in \mathcal{E}(n,k)$ such that $f(S'(B_{nyj})) = f(S(B_{nyj})) - f(E_k)$ and $V'_{km\alpha} \subset f(S'(B_{nyj}))$. Let

$$U_{km\alpha} = f(S(B_{nyj})) \cap O_{km\alpha} \cap V_{km\alpha}.$$

Then $U_{km\alpha}$ is an open set in Y. Let

$$\mathcal{U}(k,m) = \{U_{km\alpha} : \alpha \in \Lambda(k,m)\}.$$

Then $\mathcal{U}(k,m)$ is a discrete collection since $V'_{km\alpha} \subset U_{km\alpha} \subset V_{km\alpha}$ for each $V_{km\alpha} \in \mathcal{V}_{km}$ and $\mathcal{V}_{km}$ is discrete. Let $E(n,k) = \cup \mathcal{E}'(n,k)$ and $\mathcal{U}(n,k) = \cup_m \mathcal{U}(k,m)$. To complete the induction on k, take an $f^{-1}(y) \subset E(n,k)$. Then there exits a $V'_{km\alpha}$ in $\mathcal{V}'_{km}$ with $y \in V'_{km\alpha} \subset f(S'(B_{nyj}))$ by the definition of quasi-base refinement $\mathcal{V}'_k = \cup_m \mathcal{V}'_{km}$. Then we have $f^{-1}(y) \subset f^{-1}(U_{km\alpha}) \subset S(B_{nyj})$ since $V'_{km\alpha} \subset U_{km\alpha} \subset V_{km\alpha} \cap f(S(B_{nyj}))$. Then $f^{-1}(U_{km\alpha}) \subset S(B_{nyj}) \subset B_{nyj} \subset B_{ny}$ since $\mathcal{B}_{nyj} \subset \mathcal{B}_{ny}$.

Now we prove $Y = \cup_k E(n,k)$. To do it take an $f^{-1}(y) \subset Y$. Then $f^{-1}(y) \subset B_{ny} = \cup \mathcal{B}_{ny}$. Let k be the least number such that there exits a $\mathcal{B}_{nyj} \in \mathbb{B}_{ny}$ with $|\mathcal{B}_{nyj}| = k$. Then $f^{-1}(y) \cap E_k = \emptyset$ by the definition of E_k. Then $S'(B_{nyj}) \in \mathcal{E}'(n,k)$ by the definition of $\mathcal{E}'(n,k)$. So $f^{-1}(y) \subset S'(B_{nyj}) \subset E(n,k)$. □

Note 1. Let (X,τ) be a paracompact σ-space and $f : (X,\tau) \to (Y,\tau_Y)$ be a perfect mapping. Then (Y,τ_Y) is a paracompact σ-space by Corollary 4.12 in [4]. Then both (X,τ) and (Y,τ_Y) are submetrizable spaces by Theorem 4.6 and Corollary 2.9 in [4].

Theorem 2.8. *Let (X,τ) be a paracompact σ-space and $f : (X,\tau) \to (Y,\tau_Y)$ be a perfect mapping. Then both (X,τ) and (Y,τ_Y) can be submetrized by metric ρ and d respectively such that $f : (X,\tau_\rho) \to (Y,\tau_d)$ is a perfect mapping.*

Proof. Let (X, τ'_ρ) be a metrizable topology with $\tau'_\rho \subset \tau$ and (Y, d') be a metrizable topology with $d' \subset \mathcal{U}$. Let $\mathcal{B} = \cup_n \mathcal{B}_n$ be a fine-base of τ'_ρ and $\mathcal{U}' = \cup_{nk}\mathcal{U}(n,k)$ be a σ-discrete collection of open sets in Y satisfying Proposition 2.6. Then, by Lemma 2.21 in [8], there is a metrizable topology $d \subset \mathcal{U}$ on Y such that $d' \cup \mathcal{U}' \subset d$ and d is generated by the subbase $d' \cup \mathcal{U}'$ (note the last two sentences of the Proof of the Lemma 2.21 in [8]). Let $\mathcal{W} = \cup_m \mathcal{W}_m$ be a σ-discrete base of d. Then $f^{-1}(\mathcal{W}) = \{f^{-1}(W) : W \in \mathcal{W}\}$ is a σ-discrete collection of open sets with $f^{-1}(\mathcal{W}) \subset \tau$. Then, by Lemma 2.21 in [8], there is a metrizable topology $\tau_\rho \subset \tau$ on X such that $\tau'_\rho \cup f^{-1}(\mathcal{W}) \subset \tau_\rho$ and ρ is generated by the subbase $\tau'_\rho \cup f^{-1}(\mathcal{W})$. Let $\mathcal{B}_{nm} = \{B \cap f^{-1}(W) : B \in \mathcal{B}_n \text{ and } W \in \mathcal{W}_m\}$.

1) $\cup_{nm}\mathcal{B}_{nm}$ is a base of ρ.

To see it let $\mathcal{B}'$ be the collection of all finite intersections of $\tau'_\rho \cup f^{-1}(\mathcal{W})$. It is easy to check $\mathcal{B}' = \cup_{nm}\mathcal{B}_{nm}$. In fact, Let $O \in \tau_\rho$ and $x \in O$. Then there is a finite collection $\mathcal{G} = \{G_i : i \leq n(O)\} \subset \tau'_\rho$ and a finite collection $\mathcal{E} = \{f^{-1}(W_i) : i \leq m(O)\} \subset f^{-1}(\mathcal{W})$ with $x \in (\cap\mathcal{G}) \cap (\cap\mathcal{E}) \subset O$ by the definition of subbase. Then there is a $B \in \cup_n \mathcal{B}_n$ with $x \in B \subset \cap\mathcal{G}$ since $\cup_n\mathcal{B}_n$ is a base of τ'_ρ and a $W \in \cup_m \mathcal{W}_m$ with $x \in f^{-1}(W) \subset \cap\mathcal{E}$ since $\cup_m\mathcal{W}_m$ is a base of d. Then $x \in B \cap f^{-1}(W) \subset O$ and $B \cap f^{-1}(W) \in \mathcal{B}_{nm}$.

2) $f : (X, \tau_\rho) \to (Y, \tau_d)$ is continuous since $\cup_m f^{-1}(\mathcal{W}_m) \subset \tau_\rho$.

3) $f : (X, \tau_\rho) \to (Y, \tau_d)$ is ρ-compact since $\tau_\rho \subset \tau$.

4) $f : (X, \tau_\rho) \to (Y, \tau_d)$ is closed.

To prove it take an $O \in \tau_\rho$ and an $x \in S(O)$ if $S(O) \neq \emptyset$. Let $f(x) = y$. Then $f^{-1}(y) \subset S(O) \subset O$. Pick a $t \in f^{-1}(y)$. Then there is a $B_t \cap f^{-1}(W_t) \in \cup_{nm}\mathcal{B}_{nm}$ with $t \in B_t \cap f^{-1}(W_t) \subset O$ by the above 1. Then $f^{-1}(y) \subset \cup_{i\leq m}(B_i \cap f^{-1}(W_i)) \subset O$ by the above 3. Let $W'' = \cap_{i\leq m} W_i$ and $B = \cup_{i\leq m} B_i$. Then $B \in \tau'_\rho$, $W'' \in \mathcal{W}$ and $f^{-1}(y) \subset B \cap f^{-1}(W'') \subset O$. Note that $f^{-1}(y) \subset B$ implies $\tau'_\rho(f^{-1}(y), X - B) = r > 0$ since $f^{-1}(y)$ is τ'_ρ-compact and $X - B$ is τ'_ρ-closed. Then there is an n with $f^{-1}(y) \subset B_{ny} \subset B$ since $\tau'_\rho(B_{n\beta}) < 1/n$ for each $B_{n\beta} \in \mathcal{B}_n$ by 2 of fine-base of metric τ'_ρ. Then, by Proposition 2.6, there is a k and an $f^{-1}(U_{km\alpha}) \in f^{-1}(\mathcal{U}(n,k))$ with $f^{-1}(y) \subset f^{-1}(U_{km\alpha}) \subset B_{ny}$. Then we have $f^{-1}(y) \subset f^{-1}(U_{km\alpha}) \cap f^{-1}(W'') \subset B \cap f^{-1}(W'') \subset O$. Then $f^{-1}(y) \subset f^{-1}(U_{km\alpha}) \cap f^{-1}(W'') \subset S(O)$. This implies $S(O) \in \tau_\rho$ and $f(S(O)) \in d$. Then $f : (X, \tau_\rho) \to (Y, \tau_d)$ is closed by Lemma 2.2. □

Denote $\cup\{U \in \mathcal{U} : x \in U\}$ by $st(x,\mathcal{U})$. Recall that X is said to have a G_δ*-diagonal* in [5] if and only if $\{(x,x) : x \in X\} = \cap\{U_n : n = 1,2,...\}$ where U_n are open sets in the Tychonoff product $X \times X$ and recall a sequence $\{\mathcal{U}_n : n = 1,2,...\}$ of open cover of X a G_δ*-diagonal sequence* for X if $\{x\} = \cap_n st(x,\mathcal{U}_n)$ for each $x \in X$.

Note 2. In Theorem 2.8, the condition "σ-spaces" cannot be deleted since Popov in [7] gave a counterexample (Example 1) which shows that there exists a hereditarily paracompact space X with a G_δ-diagonal and a perfect map $f : X \to Y$ such that Y does not have G_δ-diagonal.

In fact, Popov proved that X have G_δ-diagonal. Then, by Theorem 2.5 in [4], X is submetrizable. Also Popov proved that Y does not have G_δ-diagonal sequence. On the one hand, by Theorem 2.2 in [4], Y does not have G_δ-diagonal. On the other hand, by Theorem 2.5 in [4], Y is not submetrizable. It explains the perfect image of a hereditarily paracompact submetric space need not to be submetrizable.

Note 3. In Theorem 2.8, when the condition "perfect map" is strengthened to "perfect open map", Chigogidze in [2] proved the following Proposition which gives a positive answer to a question of Martin [5].

Proposition. If (X,τ) admit a continuous bijection onto a metric space of weight$\leq \tau$, then its perfect open image also admits a continuous bijection onto a metric space of weight$\leq \tau$.

Related the above Proposition, Proposition 2.6 and Theorem 2.8 with their proofs, we would like to repose the following question.

Question 2.9. *(Matin* [5]*). If (X,τ) is submetric and $f : (X,\tau) \to (Y,\tau_Y)$ is a perfect mapping such that the family of nontrivial fibers of f is discrete, must Y be submetric?*

Here fiber $f^{-1}(y)$ is said to be *nontrivial* provided that the $f^{-1}(y)$ contains more than one point.

Also Proposition 2.6 and Theorem 2.8 raise the following question.

Question 2.10. *Let both (X,τ) and (Y,τ_Y) be paracompact submetric spaces and $f : (X,\tau) \to (Y,\tau_Y)$ be a perfect mapping. Then can (X,τ) and (Y,τ_Y) be submetrized by metric ρ and d respectively such that $f : (X,\tau_\rho) \to (Y,\tau_d)$ is a perfect mapping?*

3. Acknowledgements.

The author should like to express his appreciation to the referee for suggesting how best to improve this paper and numerous helpful comments.

References

[1] A.V. Arhangel'skii, *On a class of spaces containing all metric spaces and all locally bicompact spaces,* Dokl. Akad. Nauk SSSR **151** (1963) 751–754.

[2] A. Chigogidze, *The contractions of topological spaces,* Soobshch. Akad. Nauk Gruzin. SSR **92** (1978), no.1, 45–47.

[3] R. Engelking, *General topology*, Mathematical Monographs, Vol. 60, Polish Scientific Publishers, Warsaw, 1977.

[4] G. Gruenhage, *Generalized metric spaces,* Handbook of Set-Theoretic Topology, K. Kunen and J.E. Vaughan, editors, 423–501. North-Holland, Amsterdam, 1984.

[5] Harold W. Martin, *Contractibility of topological spaces onto metric spaces,* Pacific J. Math. **61** (1975), no. 1, 209–217.

[6] A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces,* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **9** (1968) 236–254.

[7] V. Popov, *A perfect map need not preserved a G_δ-diagonal,* General Topology Appl., **7** (1977) 31–33.

[8] K. Tamano, *Generalized metric spaces. II*, Topics in general topology, 367–409, North-Holland Math. Library, 41, North-Holland, Amsterdam, 1989.

School of Mathematics and LPMS, Nankai University, Tianjin 300071, P. R. China

E-mail address: hpchen@nankai.edu.cn