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by

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SELECTIVE SEPARABILITY AND SS$^+$

DOYEL BARMAN AND ALAN DOW

Abstract. Inspired by the paper [4], we continue the study of the notion of selective separability which was introduced by Scheepers in [8]. It is shown that separable Fréchet spaces are selectively separable and that it is consistent that the product of such spaces need not be. We also introduce a notion stronger than selectively separable and, motivated by the questions in [4], consider these properties in countable dense subsets of uncountable powers.

1. Introduction

The notion of selective separability (or SS) was introduced by Marion Scheepers [8] and is defined in Definition 2.1. Particularly notable is the naturalness of the SS notion in function spaces with the pointwise convergence topology, namely $C_p(X)$ for metric spaces X. For a space X, $C_p(X)$ is the subspace of $\mathbb{R}^X$ consisting of the continuous functions on X (i.e., $C(X)$ with the topology of pointwise convergence). We will let $C_p(X,2)$ be the subspace of $C_p(X)$ consisting of the 2-valued functions. Since such spaces are dense in the product space 2^X, it is also natural to consider other countable dense subsets of such powers.

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Many interesting results and questions were presented in the paper [4] and we consider some of them here. We show that every separable Fréchet space is SS. We prove that there is a dense subspace of 2^{ω_1} which is SS. We exploit the connections found in [8, 4] between the Menger Property of a space X and selective separability of $C_p(X)$. For several of our results we require extra set-theoretic hypotheses. Using $\mathsf{MA}_{\rm ctble}$ we establish that the product of two countable SS spaces may not be SS and that there is a maximal regular SS space. We seem to require CH to prove that the product of two countable Fréchet spaces may not be SS. We show that adding Sacks reals can destroy SS property. We study a stronger notion called SS^+ and show that SS does not imply SS^+.

The assumption $\mathsf{MA}_{\rm ctble}$ is the statement that the well-known statement of Martin's Axiom holds for countable posets (rather than necessarily all ccc posets). This is equivalent to the statement that the real line cannot be covered by a family of fewer than $\mathfrak{c}$ many nowhere dense sets and is known to imply that the dominating number $\mathfrak{d}$ is $\mathfrak{c}$. The bounding number $\mathfrak{b}$ is the minimum cardinality of a subset of ω^ω which has no mod finite upper bound. The pseudointersection number, $\mathfrak{p}$, is the minimum cardinality of a free filter base on ω for which there is no infinite set which is mod finite contained in each member of the filter.

2. On selective separability

Let us start this section with the definition of selective separability of a topological space X.

Definition 2.1. [8] A space X is called selectively separable (or SS) if for each sequence $\{D_n\}_n$ of dense sets, there is a selection $\{E_n \in [D_n]^{<\omega}\}_{n\in\omega}$ with dense union.

Now we have some results concerning π-weight of a space and its relation with selective separability. Before citing those results let us recall the definition.

Definition 2.2. Let (X, τ) be a topological space. A family $\zeta \subset \tau$ is a π-base of X if for each $U \in \tau$, there is a $B \in \zeta$ such that $B \subset U$.

The cardinal $\pi w(X)$ (called the π-weight of X) is the minimal cardinality of a π-base of the space X. The following pair of results are already known.

Proposition 2.3. [4] *Each space with countable π-weight is selectively separable.*

Proposition 2.4. [8] *Each countable space with π-weight $< \mathfrak{d}$ is selectively separable.*

Proof. Let us fix a sequence of indexed dense sets $\{D_n = \{d(n,l) : l \in \omega\} : n \in \omega\}$. Fix a π-base $\mathcal{U}$ of cardinality less than $\mathfrak{d}$. For each $U \in \mathfrak{U}$ there is a function $f_U \in \omega^\omega$ satisfying, for each $n \in \omega$, $U \cap \{d(n,\ell) : \ell < f_U(n)\} \neq \emptyset$. Since $| \mathfrak{U} | < \mathfrak{d}$, there is a function $g \in \omega^\omega$ such that $f_U \not<^* g$ for all U. Now let $E_n = \{d(n,l) : l < g(n)\}$. Then it is easy to check that, for each $U \in \mathcal{U}$, $U \cap E_n \neq \emptyset$ for all such n such that $f_U(n) < g(n)$, and so $U \cap \bigcup_n E_n$ is not empty. □

A space is said to be *crowded* if it has no isolated points. For convenience we will often assume that the spaces under discussion are crowded. Spaces which are not crowded are easily handled by the following observation.

Lemma 2.5. *A space X is SS if and only if the set I of isolated points is countable and $X \setminus \overline{I}$ is SS.*

We recall, and generalize, the notion of countable fan-tightness.

Definition 2.6. A space X has countable (dense) fan-tightness at $x \in X$, if for each sequence (of dense sets) $\{Y_n\}_n$ with $x \in \bigcap_n \overline{Y}_n$, there is a selection $\{W_n \in [Y_n]^{<\omega} : n \in \omega\}$ such that $x \in \overline{\bigcup_n W_n}$. A space X has countable (dense) fan-tightness if it has countable (dense) fan-tightness at each point $x \in X$.

It is immediate that each SS space has countable dense fan-tightness, but it is useful to make note of the partial converse.

Lemma 2.7. *For a space X, the following conditions are equivalent:*

(1) *X is SS,*
(2) *X is separable and has countable dense fan-tightness,*
(3) *X has countable dense fan-tightness at each point of some countable dense subset.*

Proof. It suffices to prove that condition 3 implies condition 1. We may assume that the space X is crowded. Let $\{A_n : n \in \omega\}$ be a partition of ω into infinite sets. Let $D = \{d_n : n \in \omega\}$ be a dense subset of X such that X has countable dense fan-tightness at each $d \in D$. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets of X. Now for each $n \in \omega$ and for each $k \in A_n$ we use countable fan-tightness to select $\{F^n_k : n \in \omega, k \in A_n\}$ so that $d_n \in \overline{\bigcup_{k \in A_n} F^n_k}$. Since $X = \overline{\{d_n : n \in \omega\}}$ and $\{d_n : n \in \omega\} \subseteq \overline{\bigcup_n \bigcup_{k \in A_n} F^n_k}$, we have $X = \overline{\bigcup_n \bigcup_{k \in A_n} F^n_k}$. □

One of our main results shows the surprising connection between the Fréchet property and selective separability. Let us recall the definition of a Fréchet space:

Definition 2.8. A space is called *Fréchet* if it is the case that a point is in the closure of a subset of X iff there is a sequence from the set converging to that point.

Theorem 2.9. *Each separable Fréchet space is selectively separable.*

Proof. We may assume that the space X is crowded. Let D be the postulated countable dense subset of X and let $d \in D$. By Lemma 2.7, it suffices to show that X has countable dense fan-tightness at (each point) d (of D). Since $d \in \overline{X \setminus \{d\}}$ there exists a sequence $\{d_n : n \in \omega\} \subseteq X \setminus \{d\}$ which converges to d. Let $\{D_n : n \in \omega\}$ be a family of dense subsets of X. If we replace each D_n by $\bigcup_{k \geq n} D_k$, we may assume that the sequence $\{D_n : n \in \omega\}$ is descending. For all n, $d_n \in \overline{D}_n$, which implies that there exists a sequence $S_n \subseteq D_n$ which converges to d_n. Now we observe that $d \in \overline{\bigcup_n S_n}$ and, therefore, we can select a sequence $S_d \subseteq \bigcup_n S_n$ such that $S_d \to d$. Now for all $n \in \omega$, $S_d \cap S_n$ is finite since S_d and S_n converge to distinct points. Let $F_n = S_d \cap S_n$, which is a finite subset of D_n. Now $S_d = \bigcup_n F_n$ and $d \in \overline{S}_d$, which implies $d \in \overline{\bigcup_n F_n}$. Therefore, by Lemma 2.7, X is selectively separable. □

We present the following example because it seems to us to be a very natural example of a countable space with minimal π-weight (namely $\mathfrak{d}$) which fails to be selectively separable. An example using $C_p(X)$ theory was given in [4].

Example 2.10. Consider the box topology on the countable power $(\omega+1)^\omega$ where $\omega+1$ is the usual compact ordinal topology. Let $S = \{f \in \Box(\omega+1)^\omega : (\exists n)\ f(k) = \omega \text{ iff } k \geq n\}$.

Let $D_n = \{f \in S : f(k) \neq \omega \forall k \leq n\}$. It is easily seen that D_n is a dense subset of S which is moreover open. We will show that the sequence $\{D_n : n \in \omega\}$ is a witness to the fact that S is not SS. Assume that $F_n \in [D_n]^{<\omega}$ for each $n \in \omega$. Define a function $h \in \omega^\omega$ so that $f(n) < h(n)$ for each $f \in F_n$. Now the basic open set $\Pi_{k\in\omega}[h(k), \omega]$ in $\Box(\omega+1)^\omega$ does meet S but it is clearly disjoint from $\bigcup_n F_n$. Therefore S is not selectively separable.

To show $\pi w(S) = \mathfrak{d}$, let $\mathcal{D} \subset \omega^\omega$ be a dominating family of functions of cardinality $\mathfrak{d}$. Then the basic open sets are of the form: $W(t,f) = \Pi_{i<dom(t)}\{t(i)\} \times \Pi_{i\geq dom(t)}[f(i), \omega]$, $t \in \omega^{<\omega}$ and $f \in \omega^\omega$. For any open $U(s,g)$ we can take $W(s,f) \subset U(s,g)$ where f dominates g, $f \in \mathcal{D}$. Let $\kappa < \mathfrak{d}$, then for $\{f_\alpha : \alpha < \kappa\}$, $\exists g$ such that $\mid \{n : f_\alpha(n) < g(n)\} \mid = \omega$. Then $U_\alpha \not\subset W(\emptyset, g)$, which shows that $\{U_\alpha : \alpha < \kappa\}, \kappa < \mathfrak{d}$ is not a π-base. Therefore $\pi\omega(S) = \mathfrak{d}$.

The elegant and natural connections between properties of a space X and the selective separability of its function space $C_p(X)$ was discovered in [8] and explored further in [4]. The connection is the Menger Property.

Definition 2.11. A space X has the Menger Property (or is Menger) if for each sequence $\{U_n\}_n$ of open covers, there is a selection $\{W_n \in [U_n]^{<\omega}\}_n$ such that $\bigcup_n (\bigcup W_n)$ is a cover.

For example, any σ-compact space, such as $\mathbb{R}$ or 2^ω, has the Menger Property but it is known that $\omega^\omega \approx \mathbb{R} \setminus \mathbb{Q}$ does not.

Theorem 2.12. [8, 4] *For a space X, $C_p(X)$ is selectively separable if and only if $C_p(X)$ is separable and X^n is Menger for each $n \in \omega$.*

The following theorem is due to Arhangelskii.

Theorem 2.13. [1] *X^n is Menger for each n if and only if $C_p(X)$ has countable fan tightness.*

We shall need one direction of the above result, so we include a proof for the reader's convenience.

Proposition 2.14. *If a space X has the property that X^n is Menger for each n, then $C_p(X)$ has countable fan tightness.*

Proof. Since $C_p(X)$ is homogeneous, it suffices to show that $C_p(X)$ has countable fan-tightness at the constant zero function $\underline{0}$. Let $\{D_n\}_n$ be the sequence of sets each with the constant $\underline{0}$ function as a $C_p(X)$-limit. For each n, let U_n be the collection of open sets $\{(d^{-1}(-\frac{1}{n}, \frac{1}{n}))^k : d \in D_n, k \leq n\}$. We show that U_n contains an open cover of X^k for each $k \leq n$. Fix any $k \leq n$ and $\langle x_i \rangle_{i<k} \in X^k$. Since $\underline{0}$ is a limit of D_n, there exists a $d \in D_n$ such that $d(x_i) \in (-\frac{1}{n}, \frac{1}{n})$ for each $i < k$. This, in turn, means that $\langle x_i \rangle_{i<k} \in (d^{-1}(-\frac{1}{n}, \frac{1}{n}))^k$ which is a member of U_n. Thus it follows that U_n contains an open cover of X^k. Applying the Menger Property (for X^k for each k and open covers $\{U_n : k \leq n \in \omega\}$) we can select $E_n \in [D_n]^{<\omega}$ for each n so that the finite subcollection W_n, of U_n we get from the elements $d \in E_n$ yields a cover of each X^k. In fact, we can, and do, ensure that for each $k < n$, the collection $\bigcup_{n \leq m} W_m$ contains a cover of X^k. To show that $\underline{0}$ is a limit of $\bigcup_n E_n$, let us fix any k, $\{x_i : i < k\} \subset X$ and $n \geq k$. Now we need an $e \in \bigcup_{n \leq m} E_m$ such that $e(x_i) \in (-\frac{1}{n}, \frac{1}{n})$ for $i < k$. Since $\langle x_i \rangle_{i<k}$ is covered by the collection $\bigcup_{n \leq m} W_m$, we get one such e in $\bigcup_{n \leq m} E_n$. □

These next results, also from [4], reveal some of the interesting behavior of SS in products and subspaces.

Corollary 2.15. $2^{\mathfrak{c}}$ *has a dense selectively separable subspace, namely* $C_p(2^\omega, 2)$.

Proof. Countable fan-tightness is easily seen to be hereditary and $C_p(2^\omega, 2)$ is separable. Therefore it is SS. It is well-known that $C_p(2^\omega, 2)$ is dense in 2^{2^ω}. □

Similarly we have the existence of a countable dense non-SS subspace.

Corollary 2.16. $2^{\mathfrak{c}}$ *has a countable dense non-selectively separable subspace, namely* $C_p(\omega^\omega, 2)$.

Let us mention here that G. Gruenhage [7] has established the non-trivial fact that a finite union of SS spaces is again SS. On the other hand, it is interesting to note that the union of the two countable dense subsets of the product space $2^{\mathfrak{c}}$ results in a countable space which is not SS and yet which has a dense SS subset. Certainly a countable discrete space is SS, hence the continuous image of an SS space need not be SS. A more revealing example of this is to consider a dense copy, X, of the irrationals in 2^ω, and to observe that $\{f \restriction X : f \in C_p(2^\omega, 2)\}$ is a continuous image (by the projection map from 2^{2^ω} onto 2^X) of the SS space $C_p(2^\omega, 2)$ which is itself not SS (we omit the proof). Similarly, as noted in [4], the non-SS space $C_p(\omega^\omega, 2)$ has a countable dense SS subspace consisting of those functions which are continous with respect to a coarser (compact Hausdorff) topology on ω^ω.

The following result was shown to hold for countable π-weight in [4].

Theorem 2.17. *If X and Y are both countable, selectively separable and $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is selectively separable.*

Proof. Let $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ where $\kappa < \mathfrak{b}$ be a π-base for Y. Let $\{D_k = \{d_{k,m} : m \in \omega\} : k \in \omega\}$ be the countable sequence of dense subsets of $X \times Y$. Let π_x and π_y be the natural projection onto the spaces X and Y respectively. Now the set $G_k^\alpha = \pi_x[D_k \cap (X \times B_\alpha)]$ is dense in X. Since X is selectively separable, there is a selection $F_k^\alpha \subseteq D_k$ $(k \in \omega)$ so that $\pi_x[F_k^\alpha] \subseteq G_k^\alpha$ and $\overline{\bigcup \pi_x[F_k^\alpha]} = X$. Since F_k^α is finite, $\exists f_\alpha(k) \in \omega$ so that $F_k^\alpha \subseteq \{d_{k,m} : m < f_\alpha(k)\}$. Therefore we have a sequence $\{f_\alpha : \alpha < \kappa\}$ where $f_\alpha : \omega \to \omega$. Since $\kappa < \mathfrak{b}$, there exists a function $f \in \omega^\omega$ such that $\forall \alpha < \kappa$, $f_\alpha <^* f$. Let us define $F_k = \{d_{k,m} : m < f(k)\} \subset D_k$. We claim that $\overline{\bigcup_{k\in\omega} F_k} = X \times Y$. Let us choose a basic open set $U \times B_\alpha$ of $X \times Y$, then $\exists l \in \omega$ such that $\forall i > l, f(i) > f_\alpha(i)$. Since $U \cap \bigcup \Pi_x{}^{"}F_k^\alpha \neq \emptyset$, there exists a $z \in F_k$ such that $\pi_x(z) \in U \cap \bigcup \pi_x[F_k^\alpha]$, which implies that $z \in F_k \cap (U \times B_\alpha)$. Therefore $\bigcup F_k$ is dense in $X \times Y$. ☐

Another of our main results is to confirm the conjecture in [4] that SS is not productive in general.

Theorem 2.18. ($\mathsf{MA}_{\text{ctble}}$) *There exists two countable SS spaces whose product is not SS.*

Proof. Let us consider the set $\mathbb{Q} = \{q_i : i \in \omega\}$ with the standard zero-dimensional topology generated by a countable base $\mathcal{B}_0^0 = \mathcal{B}_0^1$ of clopen sets. Let τ_0^0 and τ_0^1 denote the topologies so generated.

Obviously $(\mathbb{Q}, \tau_0^0)$ and $(\mathbb{Q}, \tau_0^1)$ are SS. We will enlarge our topology in such a way that the product space $\mathbb{Q} \times \mathbb{Q}$ will not be SS. Let $\{E_n : n \in \omega\}$ be a countable family of dense sets in $\mathbb{Q} \times \mathbb{Q}$ such that E_n hits every row and column in a singleton set, in fact for any $q \in \mathbb{Q}$, $| E_n \cap [(\{q\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q\})] | \leq 1$. Moreover we ensure that for each $q \in \mathbb{Q}$, there is at most one integer n such that $E_n \cap [(\{q\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q\})]$ is non-empty. In order to ensure the product is not SS, we let $\{\langle F_n^\alpha : n \in \omega\rangle : \alpha \in \mathfrak{c}\}$ be an enumeration of all selections $\{F_n \in [E_n]^{<\omega} : n \in \omega\}$. Let $\{S_\alpha : \alpha \in \mathfrak{c}\}$ be a listing of all the countable subsets of $\mathfrak{c}$ so that for each α, $S_\alpha \subset \alpha$. Of course the family $\{Y_\alpha = \{q_i : i \in S_\alpha \cap \omega\} : \alpha \in \mathfrak{c} \text{ and } S_\alpha \subset \omega\}$ is also a listing of $\mathcal{P}(\mathbb{Q})$.

By induction on $\alpha \in \mathfrak{c}$, we define families $\langle \mathcal{B}_\beta^0 : \beta < \alpha\rangle$, $\langle \mathcal{B}_\beta^1 : \beta < \alpha\rangle$, $\langle D_\beta^0 : \beta < \alpha\rangle$, and $\langle D_\beta^1 : \beta < \alpha\rangle$ so that, for each $i \in 2$ and $\beta < \gamma < \alpha$,

(1) $\mathcal{B}_\beta^i \subset \mathcal{B}_\gamma^i$,
(2) $\mathcal{B}_\beta^i$ has cardinality at most $|\beta + \omega|$ and is a base of clopen sets for a topology, τ_β^i, on $\mathbb{Q}$,
(3) $\{D_\xi^i : \xi < \beta\}$ is a family subsets of $\mathbb{Q}$ which are dense in the τ_β^i topology,
(4) for each n, E_n is dense in the product topology $\tau_\beta^0 \times \tau_\beta^1$, and $\bigcup_n F_n^\beta$ is not dense in the product $\tau_\gamma^0 \times \tau_\gamma^1$,
(5) if $S_\beta \subset \omega$ and Y_β is dense in $(\mathbb{Q}, \tau_\beta^i)$, then $D_\beta^i = Y_\beta$,
(6) if S_β is infinite and not contained in ω, then there is a sequence $\{E_\xi^i \in [D_\xi^i]^{<\omega} : \xi \in S_\beta\}$ such that $D_\beta^i = \bigcup_{\xi \in S_\alpha} E_\xi^i$.

To complete the $\alpha = 0$ stage of the induction, we may let $D_0^0 = D_0^1$ be any dense subset of $\mathbb{Q}$ (with the usual topology). Now we assume that $\alpha > 0$. If α is a limit and $i \in 2$, then $\mathcal{B}_\alpha^i = \bigcup_{\beta<\alpha} \mathcal{B}_\beta^i$. If α is a successor, we define $\mathcal{B}_\alpha^0$ and $\mathcal{B}_\alpha^1$ below.

The choices of D_α^0 and D_α^1 do not depend on whether or not α is a limit. If S_α is finite, then $D_\alpha^i = D_0^i$ for each $i \in 2$. If S_α is a subset of ω, then, independently for $i \in 2$, we set $D_\alpha^i = Y_\alpha$ if Y_α is dense in τ_α^i, and otherwise, let $D_\alpha^i = D_0^i$. If S_α is infinite and is not a subset of ω, then, again independently for $i \in 2$, we let D_α^i be any τ_α^i-dense set satisfying the last condition. Such a set exists since τ_α^i is SS because of Lemma 2.4 and, by the hypothesis of the theorem, $\mathfrak{d} = \mathfrak{c}$.

Finally, in the case that $\alpha = \beta + 1$ we consider the construction of $\mathcal{B}^0_\alpha, \mathcal{B}^1_\alpha$ in order to satisfy condition 2. We will choose two sets A_0 and A_1 such that $(A_0 \times A_1) \cap (\cup_n F^\alpha_n) = \emptyset$. Then $\mathcal{B}^i_\alpha$ is the topology generated by $\mathcal{B}^i_\beta \cup \{A_i, \mathbb{Q} \setminus A_i\}$.

Let us consider the countable poset,

$$
\begin{aligned}
(2.1)\quad P = \{\langle a_j, b_j\rangle_{j<M} \in [\omega^2]^{<\omega} :& \\
&(\forall j < M-1)\ (a_j < a_{j+1} \text{ and } b_j < b_{j+1}), \\
&\text{and } \left(\{q_{a_j}\}_{j<M} \times \{q_{b_j}\}_{j<M}\right) \cap \bigcup_n F^\alpha_n = \phi\}\ .
\end{aligned}
$$

We will define a family of fewer than $\mathfrak{c}$ many dense subsets of P and, applying $\mathsf{MA}_{\text{ctble}}$, select a generic filter G meeting that family of dense sets. Given such a G, we let $A_0 = \{q \in \mathbb{Q} : (\exists \langle a_j, b_j\rangle_{j<M} \in G)\ q \in \{q_{a_i} : i < M\}\}$ and $A_1 = \{q \in \mathbb{Q} : (\exists \langle a_j, b_j\rangle_{j<M} \in G)\ q \in \{q_{b_j} : j < M\}\}$.

We must define dense sets to ensure that each E_l remains dense which requires considering all combinations from $\{A_0, \mathbb{Q} \setminus A_0\} \times \{A_1, \mathbb{Q} \setminus A_1\}$.

For each $B, B' \in \mathcal{B}^0_\beta \times \mathcal{B}^1_\beta$ let

$$
\begin{aligned}
(2.2)\quad D(\ell, B, B') = \{\langle a_j, b_j\rangle_{j<M} \in P : &(\exists j < M-4) \\
&(q_{a_j}, q_{b_j}) \in E_\ell \cap (B \times B') \\
&(\exists i \in (a_j, a_{j+1}))\ (q_i, q_{b_{j+1}}) \in E_\ell \cap (B \times B') \\
&(\exists i \in (b_{j+1}, b_{j+2}))\ (q_{a_{j+2}}, q_i) \in E_\ell \cap (B \times B') \\
&(\exists i \in (a_{j+2}, a_{j+3}), i' \in (b_{j+2}, b_{j+3}))\ (q_i, q_{i'}) \in E_\ell \cap (B \times B')\}\ .
\end{aligned}
$$

The special properties of the family $\{E_k : k \in \omega\}$ ensure that each $D(\ell, n, B, B')$ is a dense subset of P. To see this, fix any $p = \langle a_j, b_j\rangle_{j<M} \in P$. For each $j < M$, there are at most four points in E_ℓ which have q_{a_j} or q_{b_j} in one of their coordinates. Let E'_ℓ be E_ℓ minus these at most $4M$ many points. Since E_ℓ is $\tau^0_\alpha \times \tau^1_\alpha$-dense, there is a $(q_{a_M}, q_{b_M}) \in (E'_\ell \setminus F^\alpha_\ell) \cap (B \times B')$. Furthermore, since $(q_{a_M}, q_{b_M}) \in E_\ell$, it follows that $(\{q_{a_M}, q_{b_M}\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q_{a_M}, q_{b_M}\})$ is disjoint from E_k for all $k \neq \ell$. Therefore it follows that $\langle a_i, b_i\rangle_{i\leq M}$ is an extension of p in P. Similarly, repeat this process and choose pairs $(a_{M+j}, b_{M+j}) \in E_l \cap (B \times B')$ (for $j < 6$) with exactly the same requirements (so as to ensure no intersection with $\bigcup_n F^\alpha_n$).

The desired extension q of p which is in the set $D(\ell, n, B, B')$ is $\langle a'_j, b'_j\rangle_{j<M+4}$ where

(1) $a'_j = a_j$ and $b'_j = b_j$ for $j \leq M$,
(2) $a'_{M+1} = a_{M+2}$ and $b'_{M+1} = b_{M+1}$,
(3) $a'_{M+2} = a_{M+2}$ and $b'_{M+2} = b_{M+3}$,
(4) $a'_{M+3} = a_{M+5}$ and $b'_{M+3} = b_{M+5}$.

By suitably skipping members of $\langle a_j, b_j\rangle_{j<M+6}$ we have ensured that each of the conditions in $D(\ell, n, B, B')$ are met by one of the pairs (a_i, b_i) $(M \leq i < M+6)$.

Next, to show that each of D^0_γ and D^1_γ for $\gamma \leq \alpha$ remain dense, we define

$$(2.3)\quad D(\gamma, B, B') = \{\langle a_j, b_j\rangle_{j<M} \in P : (\exists j < M-1)(\exists i, i') \text{ such that } \{q_i, q_{a_j}\} \subset D^0_\gamma \cap B, \{q_{i'}, q_{b_{j+1}}\} \subset D^1_\gamma \cap B', i \in (a_j, a_{j+1}) \text{ and } i' \in (b_j, b_{j+1})\}\ .$$

By a similar but easier argument as above, one can show that $D(\gamma, B, B')$ is a dense subset of P.

This completes the inductive construction of the topologies $\tau^0 = \tau^0_{\mathfrak{c}}$ and $\tau^1 = \tau^1_{\mathfrak{c}}$. The family $\{E_n : n \in \omega\}$ is a family of dense subsets of the product space, and by condition 2, it is a witness to the fact that the product is not SS. Condition 5 ensures that, for each $\ell \in 2$, $\{D^\ell_\gamma : \gamma \in \mathfrak{c}\}$ lists all τ^ℓ-dense sets. Finally, condition 6 ensures that τ^ℓ is SS. □

Let us remark that we have learned that the above result has been established independently by Bella and Gruenhage. In addition, L. Babinkostova has a stronger result from CH, namely that there are spaces X, Y such that $C_p(X)$ and $C_p(Y)$ are SS but the product is not.

In light of the fact that separable Fréchet spaces are SS, it is natural to wonder if the SS property is productive if the factors are Fréchet. We will show, this time from the continuum hypothesis, that it is not. Although it is a stronger topological statement than 2.18, we include both proofs since the set-theoretic assumption is stronger and the ZFC questions remain open (see 5). Before getting to the Fréchet product question, we turn our attention to maximal spaces and will use one of the methods from these results for the Fréchet result.

Again motivated by the results in [4], we turn our attention to maximal spaces. A space is said to be *maximal* if it is crowded and it has no strictly finer crowded topology. We restrict our interest to maximal spaces which are also regular. Let us recall van Douwen's well-known result that there are regular maximal spaces. One can deduce more from his proof.

Proposition 2.19. [5] *For any countable crowded regular space X, there is a stronger regular topology on X which contains a dense subspace D which is a maximal space.*

The following result was proven from the hypothesis that $\mathfrak{d} = \omega_1$ in [4].

Theorem 2.20. *There is a countable maximal space which is not selectively separable.*

Proof. Let us start with the countable non-SS subspace $S \subset \Box(\omega+1)^\omega$ as discussed in Example 2.10. Apply Proposition 2.19 to expand the topology (on a dense subspace) D to a maximal regular topology. We check that D cannot be SS. Of course D maps continously into a dense subset of S. Although a non-SS space can have a preimage which is SS, the reason that does not happen in this example is that the dense subsets, $\{D_n\}_{n\in\omega}$, of S from 2.10 which witness that S is not SS are dense open sets. It follows then that the sequence $\{D \cap D_n\}_{n\in\omega}$ are also dense in the maximal topology. The fact that there is no appropriate dense selection of finite sets for D follows easily from the fact that no such selection exists for the coarser topology on S. $\Box$

The next two results establish that the existence of a maximal SS space is independent of ZFC.

Theorem 2.21. *It is consistent with ZFC that there is no maximal SS space.*

Proof. Assume that X is a maximal crowded SS space and assume that ω is a dense subset. Let $\mathcal{F}$ be the filter of dense open subsets of ω. Since X is a maximal space, every dense subset of X is open (see [5]), hence $\mathcal{F}$ is also the (free) filter of dense subsets of ω. Since X is SS, it follows easily then that $\mathcal{F}$ is a P-filter in the (usual) sense that if $\{F_n : n \in \omega\} \subset \mathcal{F}$, then there is an $F \in \mathcal{F}$, such that $F \setminus F_n$ is finite for all n. Such an F can be chosen simply by applying the SS property applied to the descending sequence of dense sets $\{F_0 \cap \cdots \cap F_n : n \in \omega\}$. In addition, since X is maximal (and every dense set is open), if $I \in \mathcal{F}^+$ (i.e., $I \cap F \neq \emptyset$ for all $F \in \mathfrak{F}$) then its complement is not dense, hence I must have interior in X. In $\beta\mathbb{N}$ terminology, we have shown that $\mathcal{F}$ gives rise to a ccc P-set in ω^*. That is, the subset $K = \bigcap\{F^* : F \in \mathcal{F}\}$ is a P-set in ω^* which has the ccc (in fact, it is separable). To finish the proof, we note that it was shown in [6] that it is consistent that there are no such P-sets. □

Theorem 2.22. ($\mathsf{MA}_{\mathrm{ctble}}$) *There exists a maximal SS space.*

Proof. Let us start with ω endowed with a crowded metric topology, let τ_0 be the countable base of clopen sets. Let $\{D_\alpha : \alpha < \mathfrak{c}\}$ be the listing of all dense τ_0-dense sets. Suppose that at stage α we have a zero-dimensional topology τ_α such that for each $\alpha < \mathfrak{c}$ the following conditions are satisfied,

(1) If $\beta < \alpha$ then $\tau_\beta \subset \tau_\alpha$.
(2) If $\beta < \alpha$ and $\mathcal{D}_\beta$ is dense in τ_β,they remain dense in τ_α.
(3) For $\beta < \alpha$, B_β is either open or it has an isolated point in $\tau_{\beta+1}$.

At stage α, along with τ_α we also have the listing $\{D_\beta : \beta \in \alpha\}$ of dense subsets. If we are given a countable $S_\alpha \in [\alpha]^\omega$, hence a list $\{D_n : n \in \omega\} = \{D_\beta : \beta \in S_\alpha\}$, we use $\mathsf{MA}_{\mathrm{ctble}}$ to pick a new countable dense set D_α such that D_α can be expressed as a countable union of finite sets selected from each D_α. This ensures Selective separability. Now to ensure maximality, if we are given any $B_\alpha \subset \omega$, we first assume that B_α is not currently open, then there is some $b_\alpha \in B_\alpha$ which is also in the closure of $\omega \setminus B_\alpha$ in τ_α. Let $D_\alpha =\text{int}(B_\alpha) \cup (\omega \setminus B_\alpha)$ - which is of course dense. Now we use $\mathsf{MA}_{\mathrm{ctble}}$ to partition $D_\alpha = \bigcup_n D(\alpha, n)$ into dense sets. Also let $\{b(\alpha, n) : n \in \omega\}$ be the listing of complement of D_α. By assumption, $b(\alpha, n) \in \overline{D(\alpha, n) \setminus B\alpha}$ for all n. Now let us define a countable family of disjoint sets, for each n and 0,1 $U(\alpha, n, 0) = D(\alpha, n) \cap \text{int}(B_\alpha)$ and $U(\alpha, n, 1) = \{b(\alpha, n)\} \cup D(\alpha, n) \setminus B_\alpha$. Now we add them to our topology τ_α to get to $\tau_{\alpha+1}$ and see that $U(\alpha, n, 1) \cap B_\alpha = \{(b_\alpha)\}$. So b_α becomes an isolated point of B_α. □

It will be useful to extract the following lemma from the previous proof. However we need a strengthening of it for use with Fréchet spaces. This also necessitates a strengthing of the set-theoretic assumption beyond $\mathsf{MA}_{\text{ctble}}$.

Lemma 2.23. *If X is a countable crowded space of weight less than $\mathfrak{p}$, $\mathcal{D} \subset \mathcal{P}(X)$ is a family of almost disjoint converging sequences of X, $|\mathfrak{D}| < \mathfrak{p}$, and $S \subset X$ has dense complement and is almost disjoint from each member of $\mathcal{D}$, then there is an expansion of the topology obtained by adding countably many (crowded) clopen sets, in which S is a closed nowhere dense set, and each member of $\mathcal{D}$ is again a converging sequence.*

Proof. Fix any countable subcollection $\mathcal{B}$ of clopen subsets of X which separates points (and assume that $\mathcal{B}$ is closed under the operations of complements and finite unions and intersections). We have the set S which is almost disjoint from each $D \in \mathcal{D}$ and what we want to do is to introduce new clopen sets which will preserve that each $D \in \mathcal{D}$ is converging, and which will ensure that S is closed and discrete. If S is finite there is nothing to do, so let $S = \{s_i : i \in \omega\}$ (a faithful enumeration). The plan, like in Theorem 2.22, is to produce countably many disjoint dense subsets of X. The difficulty is to ensure that the members of $\mathcal{D}$ are not split.

Define a poset P by $p \in P$ if there is an $n_p \in \omega$ and a finite sequence $\{A^p_i : i < n_p\}$ such that these sets are pairwise disjoint, and for each $i < n_p$, $s_i \in A^p_i$ is a compact subset of $\{s_i\} \cup X \setminus S$ which satisfies that for some finite set $F \subset X$, there is a finite set $\mathcal{D}' \subset \mathcal{D}$ such that $A^p_i \setminus F = \bigcup D' \setminus F$.

We define $p < q$ if $n_q \leq n_p$ and for each $i < n_q$, $A^q_i \subset A^p_i$. We show below that P is σ-centered, from which we deduce that we can find "generic" filters that meet any collection of fewer than $\mathfrak{p}$ dense sets. In particular, we see easily that for each $D \in \mathcal{D}$ and $x \in X$, $\{p \in P : (\exists i, j < n_p) x \in A^p_j \text{ and } |D \setminus A^p_i| < \omega\}$ is dense. Furthermore, for each non-empty open $U \subset X$ and each $i \in \omega$, the set $\{p \in P : A^p_i \cap U \neq \emptyset\}$ is dense. Given a filter $G \subset P$ meeting each of these dense sets, we define $A_i = \bigcup\{A^p_i : p \in G\}$ and observe that A_i will be dense and meet S at the point s_i. Furthermore, the family $\{A_i : i \in \omega\}$ will be a partition of X. It follows easily that the topology we obtain by adding each $\{A_i, X \setminus A_i\}$ to the base will be as desired. It remains only to show that P is σ-centered.

Given any $p \in P$, we may choose a finite sequence $\{B_i^p : i < n_p\}$ of pairwise disjoint members of $\mathcal{B}$ so that $A_i^p \subset B_i^p$ for each $i < n_p$. If $p, q \in P$ are such that $n_p = n_q$ and $B_i^p = B_i^q$ for each $i < n_p$, then it is easy to see that $r = \{A_i^p \cup A_i^q : i < n_p\}$ is a common extension which is again separated by the same sequence $\{B_i^p : i < n_p\}$. Clearly then the poset P is σ-centered. □

Theorem 2.24. (CH) *There exists two countable Fréchet spaces whose product may not even be SS.*

Proof. Let us start with ω as our base set and a standard countable base $\tau_0 = \sigma_0$ of clopen sets for a zero-dimensional crowded topology on ω. Choose the sequence $\{E_n : n \in \omega\} \subset \omega^2$ just as we did in Theorem 2.18. Let π_0 and π_1 denote the two coordinate projections on $\omega \times \omega$. For a set $Y \subset \omega$, define $E(Y, 0) = \pi_0[(\omega \times Y) \cap \bigcup_n E_n]$ and $E(Y, 1) = \pi_1[(Y \times \omega) \cap \bigcup_n E_n]$. Fix an enumeration $\{(x_\alpha, S_\alpha) : \alpha \in \omega_1\}$ for $\omega \times [\omega]^\omega$. We inductively choose countable bases τ_β, σ_β for crowded 0-dimensional topologies on ω. We also inductively choose families $\{Y_\beta : \beta < \alpha\}$ and $\{Z_\beta : \beta < \alpha\}$ of converging sequences with respect to the τ_α and σ_α topologies, respectively. For convenience we assume that $\lim(Y_\beta) \in Y_\beta$ and $\lim(Z_\beta) \in Z_\beta$ for each $\beta < \alpha$ (the limits are uniquely determined by the $\tau_0 = \sigma_0$ topology). Let $\{\langle F_n^\alpha : n \in \omega\rangle : \alpha \in \omega_1\}$ be an enumeration of all selections $\{F_n \in [E_n]^{<\omega} : n \in \omega\}$.

Suppose that at stage $\alpha < \omega_1$ of our induction the following conditions are satisfied for $\gamma < \beta < \alpha$:

(1) $\tau_\gamma \subset \tau_\beta$ and $\sigma_\gamma \subset \sigma_\beta$ are countable bases on ω,
(2) for each n, E_n is dense in the product topology $\tau_\beta \times \sigma_\beta$, and $\bigcup_n F_n^\gamma$ is not dense in the product $\tau_\beta \times \sigma_\beta$,
(3) Y_γ is a τ_β-converging sequence, $E(Y_\gamma, 1)$ is σ_β closed discrete, and if x_γ is a τ_β-limit of S_γ, then for some $\xi \le \gamma$, $Y_\xi \cap S_\gamma$ is infinite and $\lim(Y_\gamma) = x_\gamma$,
(4) Z_γ is a σ_β-converging sequence, $E(Z_\gamma, 0)$ is τ_β closed discrete, and if x_γ is a σ_β-limit of S_γ, then for some $\xi \le \gamma$, $Z_\gamma \cap S_\gamma$ is infinite and $\lim(Z_\gamma) = x_\gamma$,
(5) each of the families $\{Y_\xi : \xi < \beta\}$ and $\{Z_\xi : \xi < \beta\}$ are almost disjoint.

If α is a limit, then $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$, $\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta$, and all the inductive conditions are preserved. For the successor stage, i.e., $\alpha = \beta+1$, we define τ_α and σ_α as follows. We have the sequence $\{F_n = F_n^\beta : n \in \omega\} \in [E_n]^{<\omega}$. Our plan is to first choose new clopen sets A to be added to τ_α and B to be added to σ_α with the property that $A \times B$ is disjoint from each F_n.

We will define A and B by a countable induction. Let $\{\xi_k : k \in \omega\}$ be an enumeration of α. Let $\{U_j : j \in \omega\}$ enumerate a clopen base for $\tau_\beta \times \sigma_\beta$. Finally, let $\{(i_k, j_k) : k \in \omega\}$ enumerate $\omega \times \omega$. For each $n \in \omega$, we define τ_0-closed sets A_n, A_n^-, B_n, B_n^-, so that

(1) for $k < n$, $A_k \subset A_n$, $A_k^- \subset A_n^-$, $B_k \subset B_n$, and $B_k^- \subset B_n^-$,
(2) $n \subset A_n \cup A_n^-$ and $n \subset B_n \cup B_n^-$,
(3) $A_n \cap A_n^- = \emptyset$, $B_n \cap B_n^- = \emptyset$,
(4) each of A_n and A_n^- is, mod finite, equal to a finite union of members of $\{Y_{\xi_k} : k \in \omega\}$ and, for each $k < n$, Y_{ξ_k} is, mod finite, contained in one of A_n, A_n^-,
(5) each of B_n and B_n^- is, mod finite, equal to a finite union of members of $\{Z_{\xi_k} : k \in \omega\}$ and, for each $k < n$, Z_{ξ_k} is, mod finite, contained in one of B_n, B_n^-,
(6) $A_n \times B_n$ is disjoint from $\bigcup_\ell F_\ell$,
(7) each product from $\{A_n, A_n^-\} \times \{B_n, B_n^-\}$ meets $E_{i_k} \cap U_{j_k}$ for each $k < n$.

To start the induction, we can let each of A_0, A_0^-, B_0, and B_0^- be empty. Assume that $n \in \omega$ and we have chosen the sets A_n, A_n^-, B_n, and B_n^- satisfying the inductive conditions. Each of the conditions are preserved if we add the singleton n to A_n^- providing $n \notin A_n$, and similarly add n to B_n^- if $n \notin B_n$. With this possible change then we may assume that $n+1$ is a subset of each of $A_n \cup A_n^-$ and $B_n \cup B_n^-$. We begin by considering the last inductive condition. Since $E(\{\ell\} \cup Y_\xi, 1)$ and $E(\{\ell\} \cup Z_\xi, 0)$ are nowhere dense in σ_β and τ_β respectively (for all $\ell \in \omega$ and $\xi \in \alpha$), it follows that the set

$$\big((A_n \cup A_n^- \cup E(B_n, 0)) \times \omega\big) \cup \big(\omega \times (B_n \cup B_n^- \cup E(A_n, 1))\big)$$

is a nowhere dense set in the topology $\tau_\beta \times \sigma_\beta$. Since $E_{i_n} \setminus F_{i_n}$ is dense, we can choose a point $(a_n^0, b_n^0) \in U_{j_n} \cap E_{i_n} \setminus F_{i_n}$ which is not in that product. Consider any point (a_n^0, b) for $b \in B_n$. Since $a_n^0 \notin E(B_n, 0)$, it follows that $(a_n^0, b) \notin E_\ell$ for all $\ell \in \omega$. Similarly, for all $a \in A_n$, $(a, b_n^0) \notin E_\ell$ for all $\ell \in \omega$. In addition, the family $\{E_\ell : \ell \in \omega\}$ are pairwise disjoint, hence $(A_n \cup \{a_n^0\}) \times (B_n \cup \{b_n^0\})$ is disjoint from F_ℓ for all ℓ. It is routine to recursively repeat this process to similarly choose points $\{(a_n^i, b_n^i) : i < 4\} \subset E_{i_n} \cap U_{j_n}$ (so that each of $\{a_n^i : i < 4\}$ and $\{b_n^i : i < 4\}$ have four elements). It will then follow that $(A_n \cup \{a_n^0, a_n^1\}) \times (B_n \cup \{b_n^0, b_n^2\})$ will be disjoint from $\bigcup_\ell F_\ell$ (and of course that each of $(A_n \cup \{a_n^0, a_n^1\}) \cap (A_n^- \cup \{a_n^2, a_n^3\})$ and $(B_n \cup \{b_n^0, b_n^2\}) \cap (B_n^- \cup \{b_n^1, b_n^3\})$ are empty).

We next consider the converging sequence Y_{ξ_n} with limit y_n. Since $E((n \cup B_n \cup B_n^- \cup \{b_n^i : i < 4\}), 0)$ is closed discrete, there is an integer m_n so that $Y_{\xi_n} \setminus m_n$ is disjoint from $E((n \cup B_n \cup B_n^- \cup \{b_n^i : i < 4\}), 0)$. If $y_n \in A_n \cup \{a_n^0, a_n^1\}$, then we define $A_{n+1} = A_n \cup \{a_n^0, a_n^1\} \cup (Y_{\xi_n} \setminus m_n)$ and $A_{n+1}^- = A_n^- \cup \{a_n^2, a_n^3\}$. Otherwise, $y_n \notin A_n \cup \{a_n^0, a_n^1\}$, and we set $A_{n+1} = A_n \cup \{a_n^0, a_n^1\}$ and $A_{n+1}^- = A_n^- \cup \{a_n^2, a_n^3\} \cup \{y_n\} \cup (Y_{\xi_n} \setminus m_n)$. We have maintained the requirements that $A_{n+1} \times (B_n \cup \{b_n^0, b_n^2\})$ is disjoint from F_ℓ for all ℓ. We proceed similarly with Z_{ξ_n} and $z_n = \lim(Z_{\xi_n})$. There is an integer m_n' so that $Z_{\xi_n} \setminus m_n'$ is disjoint from $E((n \cup A_{n+1}), 1)$. If $z_n \in B_n \cup \{b_n^0, b_n^2\}$, then we define $B_{n+1} = B_n \cup \{b_n^0, b_n^2\} \cup (Z_{\xi_n} \setminus m_n')$ and $B_{n+1}^- = B_n^- \cup \{b_n^1, b_n^3\}$. Otherwise, $z_n \notin B_n \cup \{b_n^0, b_n^2\}$, and we set $B_{n+1} = B_n \cup \{b_n^0, b_n^1\}$ and $B_{n+1}^- = B_n^- \cup \{b_n^1, b_n^3\} \cup \{z_n\} \cup (Z_{\xi_n} \setminus m_n')$. We have maintained the requirements that $A_{n+1} \times B_{n+1}$ is disjoint from F_ℓ for all ℓ. It should be clear that A_{n+1}, B_{n+1}, A_{n+1}^-, and B_{n+1}^- meet all the inductive requirements. Let $A = \bigcup_n A_n$ and $B = \bigcup_n B_n$ (hence $\omega \setminus A = \bigcup_n A_n^-$ and $\omega \setminus B = \bigcup_n B_n^-$). We generate new topologies from $\tau_\beta \cup \{A, \omega \setminus A\}$ and $\sigma_\beta \cup \{B, \omega \setminus B\}$ which we will temporarily denote by τ_α' and σ_α'. Of course we have ensured that $A \times B$ is disjoint from $\bigcup_\ell F_\ell$ and we have preserved that each E_ℓ is dense in $\tau_\alpha' \times \sigma_\alpha'$.

Now we define Y_β and take care to ensure that $E(Y_\beta, 1)$ is closed discrete in σ_α. Before starting, we select countably many σ_α' converging sequences to temporarily add to the collection $\{Z_\xi : \xi < \beta\}$ so that for each $\ell \in \omega$ and each $(n, m) \in \omega \times \omega$, there is a sequence, $T(\ell, n, m)$, in this collection, and a function from $T(\ell, n, m)$ into E_ℓ so that the range converges to (n, m). Now choose Y_β so as to be almost disjoint from each member of $\{Y_\xi : \xi \in \beta\}$, and to be a sequence which τ_α'-converges to x_β and, if possible, is contained in S_β. By a simple inductive thinning out process of Y_β, we can additionally ensure that $T(\ell, n, m) \setminus E(Y_\beta, 1)$ is infinite for each $\ell, n, m \in \omega$ (which uses the fact that $E(\{y\}, 1)$ is finite (even a singleton) for each $y \in \omega$). Now we apply Lemma 2.23 to expand the countable base σ'_α to a countable base σ_α so as to ensure $E(Y_\beta, 1)$ is closed discrete and while preserving that each member of the collection $\{Z_\xi : \xi \in \beta\} \cup \{T(\ell, n, m) \setminus E(Y_\beta, 1) : \ell, n, m \in \omega\}$ remains converging. The existence of the converging sequences $T(\ell, n, m)$ and the fact that τ'_α is not changing, ensures that each E_ℓ is dense in $\tau'_\alpha \times \sigma_\alpha$. Next, working with the topologies τ'_α and σ_α, we repeat the process to suitably choose a σ_α converging Z_α (satisfying condition 4) so that by expanding τ'_α to a countable base τ_α, $E(Z_\alpha, 0)$ is closed discrete. This completes the inductive construction. □

It is established in [4] that it is independent of the usual axioms that 2^{ω_1} has a dense non-selectively separable subspace. On the other hand, the following result answers a natural question posed in [4]. One should recall, as noted above, that one cannot conclude that the projection of an SS subspace of $2^{\mathfrak{c}}$ will remain SS.

Theorem 2.25. 2^{ω_1} *does have a dense SS subspace.*

Proof. If $\mathfrak{b} > \omega_1$ then every countable subset of 2^{ω_1} is selectively separable. Otherwise, let $Y = \{f_\alpha : \alpha \in \mathfrak{b}\} \subset \omega^\omega$ be $<^*$-unbounded family of increasing functions. Let $Q = \{q \in (\omega+1)^\omega : q$ is monotone and is eventually equal to $\omega\}$.

Now we make use the following result from [2].

Proposition 2.26. $X = Q \cup Y$ *has all the finite powers Menger.*

Again for the reader's convenience, we include the proof. Let us define basic open sets in the product space $(\omega+1)^\omega$:

$$[s : n] = \{f \in X : s \subset f \text{ and } f(|s|) > n\} \subset (\omega+1)^\omega$$

where $s \in \omega^{<\omega}$ and $n \in \omega$.

We prove by induction on m that X^m is Menger. Given a sequence $\langle U_n \rangle_n$ of open covers of X^m by basic open sets, we may assume that each basic open subset of a member of U_n is also in U_n. Now let us define $g(\ell)$ by recursion on ℓ. Given $l = mk^3$ and i such that $l + i < m(k+1)^3$, so that $g(l+i)(i < mk)$ has been defined, set $n = g(l+i+1)$ large enough so that for each sequence $\{s_j : j < m\} \subset (g(l+i)^{<k})$ the set $[s_0 : n] \times [s_1 : n] \times \ldots \times [s_{m-1} : n] \in U_k$ (and is added to W_k). Such a value for n exists, since we are just asking for a member of U_k which contains the point $\langle x_j : j < m \rangle$ where for $j < m$, x_j is the unique member of Q extending s_j such that $x_j(|s_j|) = \omega$. Let $\bar{g}(k) = g(m(k+1)^3)$ for each k.

Let us assume that $f_{\alpha_0} \not<^* \bar{g}$ and let $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{m-1} < \mathfrak{b}$. Now fix any k so that $g(mk^3) < \bar{g}(k) < f_{\alpha_0}(k) \leq \ldots \leq f_{\alpha_{m-1}}(k)$. For each $i < mk$ and $j < m$, there is a minimal $s^i_j \subset f_{\alpha_j} \restriction k$ such that $f_{\alpha_j}(|s^i_j|) \geq n_i = (mk^3 + i)$. It follows that $\{s^i_j : j < m\} \subset (g(mk^3+i))^{<k}$ and that $[s^i_0 : n_{i+1}] \times [s^i_1 : n_{i+1}] \times \ldots \times [s^i_{m-1} : n_{i+1}] \in W_k$. Given such an i, if $\langle f_{\alpha_j}\rangle_{j<m}$ is not in $[s^i_0 : n_{i+1}] \times [s^i_1 : n_{i+1}] \times \ldots \times [s^i_{m-1} : n_{i+1}]$, then for some $j < m$, the domain of s^{i+1}_j is strictly bigger than the domain of s^i_j. As this can only happen at most mk times, there is an $i < mk$ such that $\langle f_{\alpha_j}\rangle_{j<m}$ is in $[s^i_0 : n_{i+1}] \times [s^i_1 : n_{i+1}] \times \ldots \times [s^i_{m-1} : n_{i+1}] \in W_k$. The same argument shows that any rearrangement of the order of the elements in $\langle f_{\alpha_j}\rangle \in X^m$ will be covered by a member this choice for W_k.

If follows therefore that we are able to choose the sequence $\langle W_n\rangle_n$ to be a cover of $(X \setminus \{f_\beta : \beta < \alpha_0\})^m$. It is rather immediate that a Lindelof space which is the union of fewer than $\mathfrak{b}$ many Menger subspaces is again Menger. Therefore if follows by induction on m, that the complement in X^m of $(X \setminus \{f_\beta : \beta < \alpha_0\})^m$ is Menger. This completes the proof that X^m is Menger.

Thus $C_p(X, 2)$ is a selectively separable subspace of $2^{\mathfrak{b}}$. □

3. Results on SS$^+$

Let us introduce an interesting game with the essence of selective separablity.

Definition 3.1. (1) A space has the property SS$^+$, if player II has the winning strategy for the obvious game: player I picks a dense set D_n; player II picks a finite $E_n \subset D_n$. Player II wins if $\bigcup_n E_n$ is dense.

(2) A space X has strategic countable fan-tightness at a point $x \in X$ if player II has a winning strategy for the following game: player I picks a set A_n such that $x \in \overline{A_n}$; player II picks a finite $E_n \subset A_n$. Player II wins if x is in the closure of $\bigcup_n E_n$.

Clearly each SS$^+$ space is separable and SS.

Lemma 3.2. *Any crowded SS$^+$ space has an uncountable almost disjoint family of dense subsets; thus no maximal space is SS$^+$.*

Proof. Let $\mathcal{D}$ be the collection of all dense subsets of a space which is SS$^+$ and assume, for convenience, that ω is a dense subset. Fix a winning stragegy σ for Player II; i.e., σ is a function with domain $\bigcup_n \mathcal{D}^n$, and for $\vec{D} \in dom(\sigma)$ and $D \in \mathcal{D}$, $\sigma(\vec{D}, D) \in [D]^{<\omega}$. Since we are assuming that the space is crowded, it follows that $D_k = \omega \setminus k$ is dense for any $k \in \omega$. Let $J_0 = 0$ and, recursively define for $n \in \omega$,

$$J_{n+1} = 1 + \max \bigcup \{\sigma(\langle D_{k_i} \rangle_{i<m}) : k_0 < \cdots < k_m, \text{ for } m, k_m < J_n\} .$$

Notice that if I is any infinite subset of ω, then $D_I = \bigcup_{n \in I}[J_n, J_{n+1})$ contains the union of the responses of σ for the run of the game given by the dense sets $\{D_{J_n} : n \in I\}$. Therefore D_I is dense. $\square$

We record the following observation.

Proposition 3.3. *Each space with countable π-weight is SS$^+$.*

The following theorem shows an important result about SS and SS$^+$.

Theorem 3.4. *SS does not imply SS$^+$.*

Proof. We consider the space $X = \mathbb{Q} \cup \{f_\alpha : \alpha < \mathfrak{b}\} \subset (\omega+1)^\omega$ from Proposition 2.26. Let $S = \{f\restriction_X : f \in C_P((\omega+1)^\omega, 2)\}$ which is SS. We claim that S is not SS$^+$. To prove this, let $\mathcal{D}$ be the collection of all dense subsets of X. Assume that σ is a strategy for Player II, i.e., for each sequence $\{D_0, D_1, \ldots, D_n\} \subset \mathcal{D}$, $\sigma(D_0, D_1, \ldots, D_n) \in [D_n]^{<\omega}$. Let us fix a sequence $\vec{D} = \{D_0, D_1, \ldots, D_{n-1}\} \subset \mathcal{D}$ and set $K_{\vec{D}} = \bigcap_{D \in \mathcal{D}} \left(\bigcup_{d \in \sigma(\vec{D}, D)} \overline{d^{-1}(1)} \right)$ where the closure is taken in $(\omega+1)^\omega$.

For each $y \in (\omega+1)^\omega \setminus X$ the set $D_y = \{d \in S : y \notin \overline{d^{-1}(1)}\}$ is dense in S. To see this let us fix any finite partial function s from X into 2 which defines a basic clopen set $[s] = \{d \in S : s \subset d\}$. Since $C_p((\omega+1)^\omega, 2)$ is dense in $2^{(\omega+1)^\omega}$, there is an $f \in C_p((\omega+1)^\omega, 2)$ such that $s \subset f$ and $f(y) = 0$. It follows that $d = f\restriction_X$ is in D_y, hence $[s] \cap D_y$ is non-empty. The set $K_{\vec{D}}$ is a compact subset of X, hence it is countable.

Now let us fix a countable $M \prec H_\theta$ for $\theta = 2^{\mathfrak{c}^+}$ such that σ, X, and $\mathcal{D}$ are in M. Let $x \in X \setminus M$. If $\vec{D} \in M \cap [\mathcal{D}]^{<\omega}$ then, since it is countable, $K_D \subset X \cap M$. Therefore there is some $D' \in \mathcal{D}$ such that $x \notin \bigcup_{d \in \sigma(\vec{D}, D')} d^{-1}(1)$. Since $\sigma(\vec{D}, D')$ is simply a finite subset of

D, there is some $D^* \in \mathcal{D} \cap M$ so that $\sigma(\vec{D}, D^*) = \sigma(\vec{D}, D')$. From this it follows that we may inductively choose a play of the game $\{D_0, D_1, ...\} \subset \mathcal{D} \cap M$ such that for all $\vec{D}_n = \{D_0, D_1, ..., D_n\}, x \notin \bigcup_{d \in \sigma(\vec{D}_n)} d^{-1}(1)$. Now we have ensured that if $d \in E_n = \sigma(\vec{D}_n)$ for any n, then $d(x) = 0$. Then $\bigcup_n E_n$ is not dense in S, since it misses the clopen set $\{d \in S : d(x) = 1\}$. Therefore σ is not a winning strategy, which shows that S is not SS$^+$. □

It is natural to consider which dense subsets of 2^{ω_1} are SS$^+$ and which are not. The following result provides an interesting insight from $\mathsf{MA}_{\text{ctble}}$.

Theorem 3.5. ($\mathsf{MA}_{\text{ctble}}$) *If $\omega < \kappa < \mathfrak{c}$ and D is a countable dense subset of 2^κ, then D is not SS$^+$.*

Proof. Let $\mathcal{B}$ be the Boolean algebra generated by $\{d^{-1}(0) : d \in D\} \subseteq P(\kappa)$. Now since D is dense, $\mathcal{B}$ separates points. Let us identify κ as (the fixed ultrafilters) $X \subset \mathcal{S}(\mathcal{B})$ where $\mathcal{S}(\mathcal{B})$ is the Stone space generated by $\mathcal{B}$. With this identification, we can view D as a countable dense subset of $C_p(X, 2)$; but we may also view members of D as continuous functions on all of $\mathcal{S}(\mathcal{B})$ because they do have uniquely defined continuous extensions. For all $y \in \mathcal{S}(\mathcal{B})$ we ask if $D_y = \{d : y \in \overline{d^{-1}(0)}\}$ is dense in D or not. If not then there is finite partial function $\tau_y : X \to 2$ so that the corresponding basic open set $[\tau_y] \cap C_p(X, 2)$ is disjoint from D_y, i.e., $d \supset \tau_y$ implies $d(y) = 1$. Now given a finite partial function $\tau : X \to 2$, let $Y_\tau = \{y : \tau = \tau_y\}$.

Let us assume that for some compact crowded set $K \subset \mathcal{S}(\mathcal{B})$, we have that $X \cap \overline{Y_\tau \cap K}$ is infinite. If so then $\exists x \in \overline{Y_\tau \cap K}$, $x \notin dom(\tau)$, so we can find $d \supset \tau$ such that $d \in D$ and $d^{-1}(0)$ is a neighbourhood of x, i.e., $d(x) = 0$, which implies that $d^{-1}(0) \cap Y_\tau$ is non-empty. Let $y \in d^{-1}(0) \cap Y_\tau$. Then $d \supset \tau = \tau_y$ implies $d(y) = 0$ and $d(y) = 1$ - a contradiction. It follows then that for each τ, $\overline{Y_\tau} \cap X$ is countable.

Since there are only κ many such functions τ, we can conclude from $\mathsf{MA}_{\text{ctble}}$ that there exists uncountably many $y \in \mathcal{S}(\mathcal{B}) \setminus X$ such that $y \notin Y_\tau$ for all τ. For each such y we note that the corresponding set D_y is dense.

Now let $\mathcal{D}$ be the collection of all dense subsets of $C_p(X,2)$ and assume that σ is a strategy for Player II, i.e., for each sequence $\langle D_0, D_1, \ldots, D_n\rangle \subset \mathcal{D}$, $\sigma(\langle D_0, D_1, \ldots, D_n\rangle) \in [D_n]^{<\omega}$. Consider a sequence $\vec{D} = \langle D_0, D_1, \ldots, D_{n-1}\rangle \subset \mathcal{D}$ and set,

$$K_{\vec{D}} = \bigcap_{D\in\mathcal{D}} \left(\bigcup_{d\in\sigma(\vec{D},D)} \overline{d^{-1}(0)} \right).$$

We showed above that $K_{\vec{D}} \cap X$ is countable. Now let us again fix a countable $M \prec H_\theta$ where $\theta = 2^{\mathfrak{c}^+}$ where σ, X, κ, and D are in M. Let $x \in X \setminus M$. Since it is countable, if $\vec{D} \in M$, then $(X \cap K_D) \subset X \cap M$. Arguing as in the proof of Theorem 3.4, there is a play of the game $\{D_0, D_1, ...\} \subset \mathcal{D} \cap M$ such that for all $\vec{D_n} = \{D_0, D_1, ..., D_n\}, x \notin \bigcup_{d\in\sigma(\vec{D}_n)} d^{-1}(0)$. If $d \in E_n = \sigma(\vec{D}_n)$ for any n, then $d(x) = 1$. Then $\bigcup_n E_n$ is not dense in S, since it misses the clopen set $\{d \in D : d(x) = 0\}$. Therefore σ is not a winning strategy, which shows that S is not SS$^+$. □

We noted in the previous section that if X has countable dense fan tightness, then $C_p(X)$ is SS. Here is a similar result for SS$^+$.

Theorem 3.6. *If X is σ-compact and metrizable, then $C_p(X)$ has strategic countable fan-tightness at each point.*

Proof. Let us assume that $X = \cup_{k\in\omega} X_k$, where each X_k is compact. We describe a strategy for player II in showing that $C_p(X)$ has strategic countable fan-tightness at $\underline{0}$, the constant 0 function. Given an integer k and a set $D_k \subset C_p(X)$ which has $\underline{0}$ function as a $C_p(X)$-limit. Let $U_k = \{(d^{-1}(-\frac{1}{k}, \frac{1}{k}))^k : d \in D_k\}$. Let $H \in (X_k)^k$. Since $\underline{0}$ is a limit of D_n, there exists a $d \in D_n$ such that $d(H) \subset (-\frac{1}{k}, \frac{1}{k})$. Therefore it follows that U_k contains an open cover of $(X_k)^k$. Since $(X_k)^k$ is compact, player II can select $E_k \in [D_k]^{<\omega}$ so that the finite subcollection W_k, of U_k we get from E_k yields a cover of $(X_k)^k$.

At the end of the game, to show that player II's strategy is winning we must show that $\underline{0}$ is a limit of $\bigcup_k E_k$. Let us fix any k, $\{x_i : i < k\} \subset X$ and $n \geq k$. Now we need an $e \in \bigcup_n E_n$ such that $e(x_i) \in (-\frac{1}{n}, \frac{1}{n})$ for $i < k$. We are free to make n larger and to add elements to $\{x_i : i < k\}$, so we may assume that $k = n$ and that $\{x_i : i < k\} \subset X_n$. Therefore it follows that there is an e as required in E_n. $\square$

It is routine to generalize Lemma 2.7 to obtain the following.

Corollary 3.7. *If X is σ-compact and metrizable, then $C_p(X)$ is SS^+.*

Our next example shows that it is consistent that Fréchet does not imply SS$^+$.

Example 3.8. Let $X \subset 2^\omega$ be such that $\omega_1 \le |X| < \mathfrak{p}$. Let $D = C_p(2^\omega, 2)$ Then $D' = D \restriction X \subset C_p(X)$ and $wt(D') < |X| < \mathfrak{p} < \mathfrak{d}$ which says it is SS and with the similar argument in Theorem 3.4 shows that it is not SS$^+$, whereas the space is Fréchet since it has countable tightness (indeed it is countable) and character less than $\mathfrak{p}$.

We end this section by observing that, in contrast to Theorem 3.5, the space $2^\mathfrak{c}$ contains countable dense SS$^+$ subspace, namely $C_p(2^\omega, 2)$.

4. Forcing and Selective Separability

We have seen that if $S \subset 2^\kappa$, and we force $\kappa < \mathfrak{d}$, then S becomes SS. Also if S is a countable dense in $C_p(X, 2)$, where $X = \mathbb{Q} \cup \{f_\alpha : \alpha \in \mathfrak{b}\}$, then S is SS. Hence any forcing which preserves the value of $\mathfrak{b}$ (more precisely preserving that the unbounded families of functions remain unbounded) will preserve that S is SS.

Here we can ask a question: Can we force to destroy selective separability? The answer to this question is an immediate consequence of the following result of A. Miller.

Theorem 4.1. [Miller] *If $\mathbf{x}$ is Sacks generic over $\mathbf{V}$, then in $\mathbf{V}[\mathbf{x}]$ the set $\mathbf{V} \cap 2^\omega$ does not have the Menger Property.*

Proof. Let us define $Q = \{T \subseteq 2^{<\omega}$ infinite $: \forall \sigma, \tau (\sigma \subset \tau \in T \rightarrow \tau \in T)\}$. Note that Q is a closed subspace of $P(2^{<\omega})$ (when identified as a subspace of $2^{2^{<\omega}}$) and is homeomorphic to 2^ω.

Given the Sacks real $\mathbf{x} \in 2^\omega$ and $n \in \omega$, we define in $\mathbf{V}[\mathbf{x}]$ an open cover of $Q \cap \mathbf{V}$ by $U(n,m) = \{T \in Q : \mathbf{x}{\restriction}m \notin T$ or $|\{\ell < m : \{(\mathbf{x}{\restriction}\ell)^\frown 0, (\mathbf{x}{\restriction}\ell)^\frown 1\} \subset T| \ge n+2\}$.

A Sacks real has the property that it is not a member of any ground model closed set which does not contain a perfect set. This implies that for each $T \in Q \cap \mathbf{V}$ such that $\mathbf{x}\restriction m \in T$ for all m, then the set $\{\ell : \{(\mathbf{x}\restriction\ell)^\frown 0, (\mathbf{x}\restriction\ell)^\frown 1\} \subset T\}$ is infinite. Therefore, for each n, the family $\{U(n,m) : m \in \omega\}$ is an increasing open cover of $Q \cap \mathbf{V}$.

It is well-known that the family $\mathbf{V}\cap\omega^\omega$ is dominating in $\mathbf{V}[\mathbf{x}]$ (see [9]). Therefore to show that $Q \cap \mathbf{V}$ is not SS in $\mathbf{V}[\mathbf{x}]$, we consider a strictly increasing function $g \in \omega^\omega$ from V and show there is a $T \in Q$ such that $T \notin U(n, g(n))$ for all n. To prove this it is enough to know that if, working in $\mathbf{V}$, $\mathcal{C}$ is a collection of compact perfect subsets of 2^ω with the property that each perfect set contains one, then there is some $C \in \mathcal{C}$ such that $\mathbf{x} \in \overline{C}$. Set $\mathcal{C}$ to be the collection of all perfect subsets C of 2^ω with the property that if x_0, y_0, x_1, y_1 are distinct members of C, and $\ell < m$ are minimal such that $x_0(\ell) \neq x_1(\ell)$ and $y_0(m) \neq y_1(m)$, then there is an n such that $\ell \leq g(n) < g(n+1) < m$. Given such a perfect set C, $T_C = \{t \in 2^{<\omega} : (\exists y \in C) t \subset y\}$ will be a member of Q, and the Sacks real $\mathbf{x}$ will be in $\overline{C}$ precisely if for all m, $\mathbf{x}\restriction m \in T_C$. It is routine to see that each perfect set K contains a perfect set in $\mathcal{C}$, hence there is some such C such that $\mathbf{x}\restriction m \in T_C$ for all m. The definition of $\mathcal{C}$ ensures that for each n, $\{\ell \leq g(n) : \{(\mathbf{x}\restriction\ell)^\frown 0, (\mathbf{x}\restriction\ell)^\frown 1\} \subset T_C\}$ will have cardinality less than $n+2$.

This of course completes the proof that $Q\cap\mathbf{V}$ fails to have Menger property in $\mathbf{V}[\mathbf{x}]$. □

From this result, we observe the interesting fact that there is an SS$^+$ space, namely, $S = (C_p(2^\omega, 2), \tau^{\mathbf{V}})$, for which the SS property is also destroyed by adding a Sacks real.

5. Open Problems

We have shown with the help of $\mathsf{MA}_{\text{ctble}}$ or CH that, the product of two SS spaces, even two Fréchet spaces might not be SS. So the very natural question which was asked in [4], is,

Problem 5.1 Is it true in ZFC that product of two SS spaces is not SS?

Also what happens when we consider SS$^+$? That is,

Problem 5.2 Suppose X and Y are two SS$^+$ spaces. Must $X \times Y$ or $X \cup Y$ be SS$^+$?

Problem 5.3 What happens with the product if X is SS and Y is SS^+?

We know that $C_p(\omega^\omega)$ has a dense SS subspace, we repeat the question from [4]:

Problem 5.4 If X is separable metric space, then must $C_p(X)$ have a dense SS subspace?

And also we know that if $\kappa < \mathfrak{d}$, then 2^κ has a dense SS subspace, so what happens in general?

Problem 5.5 Is there a $\kappa < \mathfrak{c}$ such that 2^κ fails to have a dense SS subspace?

Problem 5.6 Is there a ZFC example of a countable space which is Fréchet but not SS^+?

Problem 5.7 Do any of $\mathfrak{p} = \mathfrak{c}$, Martin's Axiom, or ZFC suffice to produce two countable Fréchet spaces whose product is not SS?

References

[1] A. V. Arhangel'skii, *The structure and classification of topological spaces and cardinal invariants*, (Russian), Uspekhi Mat. Nauk **33** (1978), no. 6(204), 29–84.

[2] T. Bartoszyński and B. Tsaban, *Hereditary topological diagonalizations and the Menger-Hurewicz Conjectures*, Proceedings of American Mathematical Society **134** (2006), 605–615, MR 2176030 (2006f:54038)

[3] Murray G. Bell, *On the combinatorial principle* $P(\mathfrak{c})$, Fund. Math. **114** (1981), no. 2, 149–157. MR 643555 (83e:03077)

[4] Angelo Bella, Maddalena Bonanzinga, Mikhail V. Matveev and Vladimir V. Tkachuk, *Selective Separability: General facts and behaviour in countable spaces*, Topology Proc. **32** (2008), 15–30.

[5] Eric K. van Douwen, *Applications of maximal topologies*, Topology Appl. **51** (1993), no. 2, 125–139. MR 1229708 (94h:54012)

[6] Rvszard Frankiewicz, Saharon Shelah, and Paweł Zbierski, *On closed P-sets with ccc in the space* ω^*, J. Symbolic Logic **58** (1993), no. 4, 1171–1176. MR 1253914 (95c:03125)

[7] Gary Gruenhage, *A note on Selectively Separable Spaces*, August 2008, personal communication.

[8] Marion Scheepers, *Combinatories of open covers. VI. Selectors for sequences of dense sets*, Quaest. Math. **22** (1990), no. 1, 109–130

[9] Saharon Shelah, *Proper and improper forcing.* Springer-Verlag, Berlin, second edition, 1998.

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