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TOPOLOGICAL GROUPOIDS WITH LOCALLY COMPACT FIBRES

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Abstract. For developing an algebraic theory of functions on a topological groupoid (more precisely to define convolution that gives the algebra structure on a function space associated with the groupoid), one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity".

In the group case, the existence of a positive measure, invariant under left translation (Haar measure), is equivalent with the existence of a unique locally compact topology generating the Borel structure. By analogy with the group case, it is usual to endow the groupoid with a locally compact topology. However the product, as well as the quotient topology on the principal groupoid R of a locally compact groupoid G are not necessarily locally compact topologies. Starting from the fact that the notion of Haar system has sense on a topological groupoid with locally compact fibres, the purpose of this paper is to introduce various topologies on G (not necessarily locally compact) such that the fibres of G are locally compact Hausdorff subspaces and to prove that we can endow R with similar topologies.

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1. Introduction

We shall use the definition of a topological groupoid given by Jean Renault in [3]: a *groupoid* is a set G endowed with a *product map*

$$(x,y) \mapsto xy \ \left[: G^{(2)} \to G\right]$$

where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map*

$$x \mapsto x^{-1} \ [: G \to G]$$

such that the following conditions hold:

(1) If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)\,z = x\,(yz)$.

(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.

(3) For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)\,x^{-1} = z$.

(4) For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}\,(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range map*, respectively the *source (domain) map*. It follows easily from the definition that they have a common image called the *unit space* of G, which is denoted $G^{(0)}$. Its elements are *units* in the sense that $xd(x) = r(x)\,x = x$.

The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. Also for $u, v \in G^{(0)}$, $G_v^u = G^u \cap G_v$.

If A and B are subsets of G, one may form the following subsets of G:

$$\begin{aligned} A^{-1} &= \left\{x \in G : \ x^{-1} \in A\right\} \\ AB &= \left\{xy : (x,y) \in G^{(2)} \cap (A \times B)\ \right\}. \end{aligned}$$

For each unit u, $G_u^u = \{x : r(x) = d(x) = u\}$ is a group, called *isotropy group* at u. The group bundle

$$\{x \in G : \ r(x) = d(x)\}$$

is denoted G', and is called the *isotropy group bundle* of G.

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure. This means that the inverse map $x \mapsto x^{-1}$ $[: G \to G]$ is continuous, as well as the product map $(x, y) \mapsto xy$ $\left[: G^{(2)} \to G\right]$ is continuous, where $G^{(2)}$ has the induced topology from $G \times G$.

Any groupoid G defines an equivalence relation on the unit space $G^{(0)}$. Two units $u, v \in G^{(0)}$ are equivalent is there is $x \in G$ such that $r(x) = u$ and $d(x) = v$. The graph of this equivalence relation will be denoted in this paper by

$$R = \{(r(x), d(x)),\ x \in G\}.$$

We shall also denote by $(r, d) : G \to R$, the map defined by

$$(r, d)(x) = (r(x), d(x)) \text{ for all } x \in G.$$

With the product $(u, v)(v, w) = (u, w)$ and inverse $(u, v)^{-1} = (v, u)$, R becomes a groupoid which will be called the *principal groupoid associated with* G. If G is a topological groupoid, then we can consider the subspace topology on R induced from $G^{(0)} \times G^{(0)}$ endowed with product topology. We shall call this topology the *product topology* [4, p. 5] on R. If the topology on G is locally compact, then the product topology on R is locally compact if and only if the graph of the equivalence relation is locally closed. On the other hand, we can endow R with the *quotient topology* from G as in [4]: the finest topology on R with the property that $(r, d) : G \to R$ is continuous. If the restriction of the range map to the isotropy group bundle of G is open, then the quotient topology is locally compact. But in general this is no longer true.

The purpose of this paper is to introduce various topologies on G (not necessarily locally compact) such that the fibres of G are locally compact Hausdorff subspaces and to prove that we can endow R with similar topologies. In Section 2 we start with a topological groupoid $(G,\ \tau_G)$, we introduce a topology $\tau_R(\tau_G)$ on R and a new topology $\tau_{G \vee R}$ on G such that:

(1) $(R, \tau_R(\tau_G))$ is a topological groupoid and $\tau_R(\tau_G)$ is finer that the quotient topology on R.
(2) $(G, \tau_{G \vee R})$ is a topological groupoid and $\tau_{G \vee R}$ is finer than τ_G.
(3) For every $u \in G^{(0)}$, $\tau_{G \vee R}$ and τ_G induce the same subspace topologies on G^u and G_u.

(4) If $(r,d):(G,\ \tau_G)\to R$ is open with respect to the quotient topology on R, then $\tau_{G\vee R}=\tau_G$.
(5) The map $(r,d):(G,\ \tau_{G\vee R})\to(R,\ \tau_R(\tau_G))$ is an open continuous map. Consequently,
  a): the quotient topology on R with respect to this map, $\tau_R(\tau_G)$ and $\tau_R(\tau_{G\vee R})$ coincide.
  b): the subspace topology on $G^{(0)}\subset G$ with respect to $\tau_{G\vee R}$ is the same with the topology on $G^{(0)}$ viewed as a unit space of R (endowed with $\tau_R(\tau_G)=\tau_R(\tau_{G\vee R})$).
  c): if $r':G'\to G^{(0)}$, $r'(x)=r(x)$ is the restriction of the range map to G', isotropy group bundle of G, then r' is an open map, when G' and $G^{(0)}$ are endowed with the subspace topologies from G with respect to $\tau_{G\vee R}$.
  d): the range map of G endowed with $\tau_{G\vee R}$ is open if and only if the range map of $(R,\tau_R(\tau_G))$ is open.

The topological groupoids $(G,\ \tau_G)$ studied in Section 3 will have closed points, Hausdorff unit space and locally compact Hausdorff fibres. Moreover, we shall consider that each topological groupoid $(G,\ \tau_G)$ is endowed with a family $\mathcal{K}$ of subsets with similar properties as conditionally compact subsets in the sense of [4], we shall prove that R can be endowed with a similar family of subsets, and we shall introduce two new Hausdorff topologies, $\tau_{\mathcal{K}}$ and $\tau_{\mathcal{K}\vee R}$, on G, with the following properties:

(1) $\tau_{\mathcal{K}}$, τ_G induce the same subspace topology on every closed Hausdorff subspace S of G. Particularly, if G is Hausdorff, then $\tau_{\mathcal{K}}=\tau_G$.
(2) $\tau_{\mathcal{K}\vee R}$, $\tau_{G\vee R}$ induce the same subspace topology on every closed Hausdorff subspace S of G.
(3) For every $u\in G^{(0)}$, $\tau_{\mathcal{K}\vee R}$, $\tau_{G\vee R}$, $\tau_{\mathcal{K}}$ and τ_G induce the same subspace topology on G^u.
(4) If $U\in\tau_G$ is a Hausdorff set, $K\in\mathcal{K}$, $K\subset U$, then every function $f:G\to\mathbb{C}$ which is τ_G-continuous on U and vanishes outside K, is $\tau_{\mathcal{K}}$-continuous on G.

If τ_G is second countable, then

(1) The quotient topology on R and $\tau_R(\tau_G)$ generate the same Borel structures.

(2) The Borel structures on (G, τ_G) and $(G,\ \tau_{G\vee R})$ are the same (the Borel sets of a topological space are taken to be the σ-algebra generated by the open sets).

2. Topologies on a groupoid and its associated principal groupoid

Definition 2.1. Let G be a topological groupoid, τ_G be its topology and let R be the principal groupoid associated with G. If $\mathcal{F}$ is a finite collection of open subsets of G, then we define

$$\begin{aligned} \mathcal{U}(\mathcal{F}) &= \{(u,v) \in R,\ G_v^u \cap U \neq \emptyset \text{ for all } U \in \mathcal{F}\} \\ &= \bigcap_{U\in\mathcal{F}} (r,d)(U). \end{aligned}$$

It is easy to check that the sets $\mathcal{U}(\mathcal{F})$ form a basis for a topology on R. Let us call this topology the *transported topology from G* and let us denoted it by $\tau_R(\tau_G)$.

Remark 2.2. If G is a topological groupoid and R is the principal groupoid associated with G, then:

(1) The transported topology from G on R is finer than the quotient topology on R which is finer than the product topology on R. Indeed, let D be an open subset of R with respect to the quotient topology on R. Then $(r,d)^{-1}(D)$ is an open subset of G, and since

$$\mathcal{U}\left(\left\{(r,d)^{-1}(D)\right\}\right) = (r,d)\left((r,d)^{-1}(D)\right) = D,$$

it follows that D is open with respect to transported topology from G.

(2) If $(r,d) : G \to R$ is an open map with respect to the quotient topology on R, then the transported topology from G and the quotient topology on R coincide. In particular, if G is a principal groupoid, then $(r,d) : G \to R$ is a homeomorphism with respect to the quotient topology on R which coincides with the transported topology from G on R.

Lemma 2.3. *Suppose that G is a topological groupoid, and R is the principal groupoid associated with G endowed with the transported topology from G (Definition 2.1). If $(u_i, v_i) \to (u,v)$ in R, then $(u_i, v_i) \to (u,v)$ with respect to the quotient topology as well as the product topology on R.*

Proof. It easily follows from the fact that the transported topology from G on R is finer than the quotient topology on R which is finer than the product topology on R. $\square$

Remark 2.4. If G is a topological groupoid having Hausdorff unit space, and R is the principal groupoid associated with G endowed with $\tau_R(\tau_G)$, then R is Hausdorff.

Lemma 2.5. *Suppose that G is a topological groupoid, and R is the principal groupoid associated with G endowed with the transported topology from G (Definition 2.1). Let $(u_i, v_i)_{i\in I}$ be a net in R. Then $(u_i, v_i) \to (u, v)$ in R if and only if the following condition is satisfied:*

> (C)**:** *If $x \in G_v^u$, then every subnet $(u_{i_l}, v_{i_l})_l$ has a subnet $\left(u_{i_{l_j}}, v_{i_{l_j}}\right)_j$ with the property that there are $x_{i_{l_j}} \in G_{v_{i_{l_j}}}^{u_{i_{l_j}}}$ such that $x_{i_{l_j}} \to x$.*

Proof. Let us assume that $(u_i, v_i) \to (u, v)$ in R and let us prove that (C) holds. Let $x \in G_v^u$. If $(u_i, v_i) \to (u, v)$, then any subnet of $(u_i, v_i)_{i\in I}$ converges to (u, v). Therefore relabeling we may work with $(u_i, v_i)_i$ instead of $(u_{i_l}, v_{i_l})_l$. Let

$J = \left\{(i, U) : \ i \in I, U \text{ open neighborhood of } x \text{ and } G_{v_i}^{u_i} \cap U \neq \emptyset \ \right\}$.

Given (i_1, U_1) and (i_2, U_2) in J, $U_3 = U_1 \cap U_2$ is an open neighborhood of x, and $\mathcal{U}(\{U_3\})$ is a neighborhood of (u, v). Hence, there is an i, dominating both i_1 and i_2, such that $G_{v_i}^{u_i} \cap U_3 \neq \emptyset$. It follows that J is directed by $(i, U) \geq (i', U')$ if $i \geq i'$ and $U \subset U'$. For each $(i,\ U) \in J$, choose $x_{(i,U)} \in G_{v_i}^{u_i} \cap U$. Then $\left(x_{(i,U)}\right)_{(i,U)\in J}$ is a net converging to x. For each $(j, U) \in J$, let $i_{(j,u)} = j$. Then $\left(u_{i_{(j,U)}}, v_{i_{(j,U)}}\right)_{(j,U)\in J}$ is the subnet required in (C).

Now suppose that (C) holds. Let $\mathcal{F}$ be a finite collection of open subsets of G and suppose that $(u, v) \in \mathcal{U}(\mathcal{F})$. If $U \in \mathcal{F}$ and if we don't eventually have $G_{v_i}^{u_i} \cap U \neq \emptyset$, then we can pass to a subnet, relabel, and assume that $G_{v_i}^{u_i} \cap U = \emptyset$ for all i. Since $(u, v) \in \mathcal{U}(\mathcal{F})$ and $U \in \mathcal{F}$, it follows that there is $x \in G_v^u \cap U$. But if $x \in G_v^u \cap U$, then (C) implies that there is a subnet $\left(u_{i_j}, v_{i_j}\right)_j$ and there are $x_{i_j} \in G_{v_{i_j}}^{u_{i_j}}$ such that $x_{i_j} \to x$. But the x_{i_j} must eventually be in U and hence $G_{v_{i_j}}^{u_{i_j}} \cap U \neq \emptyset$. This is a contradiction and completes the proof. $\square$

Proposition 2.6. *Suppose that G is a topological groupoid, and R is the principal groupoid associated with G. If R is endowed with $\tau_R(\tau_G)$ (Definition 2.1), then R is a topological groupoid.*

Proof. If $\mathcal{U}(\mathcal{F})$ is an open neighborhood of (u,v), then $\mathcal{U}(\mathcal{F})^{-1} = \mathcal{U}\left(\mathcal{F}^{-1}\right)$ is an open neighborhood of (v,u), where

$$\mathcal{F}^{-1} = \left\{U^{-1},\ U \in \mathcal{F}\right\}.$$

Hence the inverse map of R is continuous. Let $(u_i,v_i) \to (u,v)$ and $(v_i,w_i) \to (v,w)$ in R. Let us show that $(u_i,w_i) \to (u,w)$. Let us prove that condition (C) from Lemma 2.5 holds. Let $x \in G^u_w$ and let $y \in G^u_v$. Then $y^{-1}x \in G^v_w$. If $(u_i,v_i) \to (u,v)$, then (C) implies that any subnet of $(u_i,v_i)_i$ (also denoted $(u_i,v_i)_i$) contains a subnet $\left(u_{i_j},v_{i_j}\right)_j$ with the property that there are $y_{i_j} \in G^{u_{i_j}}_{v_{i_j}}$ such that $y_{i_j} \to y$. Similarly, passing to a subnet and relabeling there are $z_{i_j} \in G^{v_{i_j}}_{w_{i_j}}$ such that $z_{i_j} \to y^{-1}x$. Thus $y_{i_j}z_{i_j} \in G^{u_{i_j}}_{w_{i_j}}$ and $y_{i_j}z_{i_j} \to x$. □

Proposition 2.7. *Suppose that G is a topological groupoid, and R is the principal groupoid associated with G. For every $u \in G^{(0)}$, let $\theta_u : G^u \to R^u$ be the map defined by $\theta_u(x) = (r,d)(x) = (u,d(x))$ for all $x \in G^u$. If R is endowed with $\tau_R(\tau_G)$ (Definition 2.1), then for every $u \in G^{(0)}$, θ_u is a continuous open map.*

Proof. Let $u \in G^{(0)}$. Let $(x_i)_i$ be a net in G^u converging to $x \in G^u$ and let us prove that $(u,d(x_i)) \to (u,d(x))$ in R with respect to transported topology from G. Let us prove that condition (C) from Lemma 2.5 holds. Let $y \in G^u_{d(x)}$. If for every i we choose $y_i = yx^{-1}x_i$, then $y_i \in G^u_{d(x_i)}$ and $y_i \to y$. Consequently, condition (C) from Lemma 2.5 holds.

Condition (C) from Lemma 2.5 implies that θ_u is an open map. Otherwise there is an open subset U of G and a net $(u,v_i)_i$ converging to an element $\theta_u(x) \in \theta_u(U \cap G^u)$ with $x \in U$ such that $(u,v_i) \notin \theta_u(U \cap G^u)$ for all i. By (C), there is a subnet $\left(u,v_{i_j}\right)_{j\in J}$ and a net $\left(x_{i_j}\right)_{j\in J}$ such that $x_{i_j} \to x$ and $x_{i_j} \in G^u_{v_{i_j}}$ for all j. Since x_{i_j} is eventually in U, it follows that $\left(u,v_{i_j}\right) = \theta_u\left(x_{i_j}\right)$ is eventually in $\theta_u(U \cap G^u)$ and that contradicts the fact that $(u,v_i) \notin \theta_u(U \cap G^u)$ for all i. □

Lemma 2.8. *Let G be a topological groupoid, $\tau_{G^{(0)}}$ be the subspace topology on $G^{(0)} \subset G$ and let R be the principal groupoid associated with G. Suppose that R is endowed with transported topology from G (Definition 2.1). If $\tau_{R^{(0)}}$ is topology induced on $G^{(0)}$ seen as a unit space of R, then $\tau_{R^{(0)}}$ is finer than $\tau_{G^{(0)}}$. A basis for $\tau_{R^{(0)}}$ is given by:*

$$\bigcap_{U\in\mathcal{F}} r\left(U \cap G'\right)$$

where $\mathcal{F}$ runs over all finite collection of open subsets of G and G' is the isotropy group bundle of G.

Consequently, if $r' : G' \to G^{(0)}$, $r'(x) = r(x)$ is the restriction of the range map to G', then r' is an open map, when G' is endowed with the subspace topology from G and $G^{(0)}$ is endowed with $\tau_{R^{(0)}}$.

Proof. A basis for the subspace topology of $diag\left(G^{(0)}\right) \subset G$ is given by:

$$\left(\bigcap_{U\in\mathcal{F}} (r,d)(U)\right) \bigcap diag\left(G^{(0)}\right) = \bigcap_{U\in\mathcal{F}} (r,d)\left(U \cap G'\right)$$

where $\mathcal{F}$ runs over all finite collection of open subsets of G. Therefore a basis for $\tau_{R^{(0)}}$ is given by:

$$\bigcap_{U\in\mathcal{F}} r\left(U \cap G'\right)$$

where $\mathcal{F}$ runs over all finite collection of open subsets of G. Since any open set with respect to $\tau_{G^{(0)}}$ is of the form $V \cap G^{(0)}$ with V an open subset of G and since

$$V \cap G^{(0)} = r\left(r^{-1}\left(V \cap G^{(0)}\right)\right) = r\left(r^{-1}(V) \cap G'\right),$$

it follows that $\tau_{R^{(0)}}$ is finer than $\tau_{G^{(0)}}$. □

Definition 2.9. Let G be a topological groupoid, τ_G be its topology and let R be the principal groupoid associated with G. Suppose that R is endowed with the transported topology from G (Definition 2.1). Let us denote by $\tau_{G\vee R}$ the least upper bound topology of $\left\{\tau_G,\ \tau^{-1}(R)\right\}$, where $\tau^{-1}(R)$ is the initial topology on G with respect to the map $(r,d) : G \to R$. Let us call $\tau_{G\vee R}$ the *modified topology on G with respect to R.*

It is easy to see that:

(1) A basis for $\tau_{G\vee R}$ is given by

$$V\bigcap\left(\bigcap_{U\in\mathcal{F}}(r,d)^{-1}\left((r,d)\left(U\right)\right)\right)$$

where V runs over all open sets of G and$\mathcal{F}$ runs over all finite collections of open subsets of G.

(2) A net $(x_i)_i$ converges to x with respect to $\tau_{G\vee R}$ if and only if $(x_i)_i$ converges to x with respect to τ_G and $(r(x_i), d(x_i))_i$ converges to $(r(x), d(x))$ in R with respect to the transported topology from G.

(3) G endowed with $\tau_{G\vee R}$ is a topological groupoid.

(4) For every $u \in G^{(0)}$, $\tau_{G\vee R}$ and τ_G induce the same subspace topology on G^u (it suffices to see that if θ_u is the map from Proposition 2.7, then $(r,d)^{-1}((r,d)(U)) \cap G^u = \theta_u^{-1}(\theta_u(U \cap G^u)))$.

Theorem 2.10. *Let G be a topological groupoid, τ_G be its topology and let R be the principal groupoid associated with G. If R is endowed with $\tau_R(\tau_G)$, the transported topology from G, (Definition 2.1), then there is a topology $\tilde{\tau}_G$ on G satisfying the following conditions:*

(1) *$\tilde{\tau}_G$ is finer than τ_G.*

(2) *G endowed with $\tilde{\tau}_G$ is a topological groupoid.*

(3) *For every $u \in G^{(0)}$, $\tilde{\tau}_G$ and τ_G induce the same subspace topologies on G^u and G_u.*

(4) *If G is endowed with $\tilde{\tau}_G$ and R is endowed with $\tau_R(\tau_G)$, then $(r,d): G \to R$ is an open continuous map. Consequently,*
   - **a):** *the quotient topology on R with respect to this map, $\tau_R(\tau_G)$ and $\tau_R(\tilde{\tau}_G)$ coincide.*
   - **b):** *the subspace topology on $G^{(0)} \subset G$ with respect to $\tilde{\tau}_G$ is the same with the topology on $G^{(0)}$ viewed as a unit space of R (endowed with $\tau_R(\tau_G) = \tau_R(\tilde{\tau}_G)$).*
   - **c):** *if $r' : G' \to G^{(0)}$, $r'(x) = r(x)$ is the restriction of the range map to G', isotropy group bundle of G, then r' is an open map, when G' and $G^{(0)}$ are endowed with the subspace topologies from G with respect to $\tilde{\tau}_G$.*
   - **d):** *the range map of G endowed with $\tilde{\tau}_G$ is open if and only if the range map of R is open.*

Proof. Let us take $\tilde{\tau}_G = \tau_{G \vee R}$, the modified topology on G with respect to R (Definition 2.9). It is easy to verify that $\tilde{\tau}_G$ satisfies the required conditions. □

Remark 2.11. The modified topology on G with respect to R (Definition 2.9) is the coarsest topology on G that satisfies the requirements in Theorem 2.10.

3. Generalized conditionally compact groupoids

A topological groupoid G is said to be locally compact if it is locally compact as a topological space (this means that every point $x \in G$ has a compact Hausdorff neighborhood). Thus any locally compact groupoid G (in the above sense) is locally Hausdorff.

The construction of the C^*-algebra of a locally compact groupoid (introduced in [3]) extends the well-known case of a group. In the case of a locally compact Hausdorff groupoid G the space $C_c(G)$ of continuous functions with compact support is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. The C^*-algebra of G is the completion of $C_c(G)$. If G is a not necessarily Hausdorff, locally compact groupoid, then as pointed out by A. Connes [1], one has to modify the choice of $C_c(G)$ (because $C_c(G)$ it is too small to capture the topological or differential structure of G). Let us assume that $G^{(0)}$ is Hausdorff and let $\mathcal{C}_c(G)$ be the space of complex valued functions on G spanned by functions f which vanishes outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. Since in a non-Hausdorff space a compact set may not be closed, the functions in $\mathcal{C}_c(G)$ are not necessarily continuous on G. According to [2, Lemma 1.3/p. 52], if $(U_i)_i$ is a covering of G by open Hausdorff subsets, then the functions in $\mathcal{C}_c(G)$ are finite sums $\sum_i f_i$, where f_i is a continuous compactly supported function on U_i. We shall introduce Hausdorff topologies on G with the property that every function f in $\mathcal{C}_c(G)$ is a bounded continuous function on G with respect to these topologies.

A *conditionally compact* subset of a topological groupoid G is a subset K such that for every compact subset L of $G^{(0)}$, $K \cap r^{-1}(L)$ and $K \cap d^{-1}(L)$ are compact subsets of G ([4]). Starting from the properties of the conditionally compact subsets in the sense of [4], we shall introduce a definition of a locally generalized conditionally compact groupoid in a more general sense.

Definition 3.1. Let G be a topological groupoid. By a *family of generalized conditionally compact subsets* of G we mean a family $\mathcal{K}$ of subsets of G satisfying the following conditions:

(1) For every $K \in \mathcal{K}$, $K^{-1} \in \mathcal{K}$.
(2) For every $K_1, K_2 \in \mathcal{K}$, there is $K_3 \in \mathcal{K}$ such that $K_1 K_2 \subset K_3$.
(3) For every $K_1, K_2 \in \mathcal{K}$, $K_1 \cup K_2 \in \mathcal{K}$.
(4) For every $u \in G^{(0)}$ and every $K \in \mathcal{K}$, $K \cap G^u$ (and hence $K \cap G_u$) is compact.
(5) For every $K \in \mathcal{K}$ and every net $(x_i)_{i \in I}$ in G converging to x, there is $i_0 \in I$ and a compact set K_0 such that $K \cap (r,d)^{-1}(L) \subset K_0 \subset K$, where $L = \{(r,d)(x_i),\ i \geq i_0\} \cup \{(r,d)(x)\}$.

Any $K \in \mathcal{K}$ will be called a *generalized conditionally compact subset* of G.

Definition 3.2. Let G be a topological groupoid such that:

(1) The points are closed in G (or equivalently, G is a T_1-space).
(2) $G^{(0)}$ is a Hausdorff subspace of G.

Then G is said to be a *locally generalized conditionally compact groupoid* with respect to $\mathcal{K}$ if $\mathcal{K}$ is a family of generalized conditionally compact subsets (in the sense of Definition 3.1) such that each point has a neighborhood basis of sets K belonging to $\mathcal{K}$.

Remark 3.3. Let G be a topological groupoid with the property that the points are closed in G. If $G^{(0)}$ is Hausdorff, then for every $u \in G^{(0)}$, G^u and G_u are Hausdorff. As in [5, Proposition 2.8/p. 569], the set

$$Z = \{(x,y) \in G^u \times G^u :\ d(x) = d(y)\}$$

is closed in $G^u \times G^u$, being the set where two continuous maps (to a Hausdorff space) coincide. Let $\varphi : Z \to G$ defined by $\varphi(x,y) = xy^{-1}$ for all $(x,y) \in Z$. Since $\{u\}$ is closed in G, $\varphi^{-1}(\{u\})$ is closed in Z. Furthermore, Z being closed in G, it follows that $\varphi^{-1}(\{u\})$, which is the diagonal of $G^u \times G^u$, is closed in G.

If G is a locally generalized conditionally compact groupoid with respect to $\mathcal{K}$, then for every $u \in G^{(0)}$, G^u and G_u are locally compact Hausdorff subspaces of G. Indeed, each point x in G^u (respectively, in G_u) has a neighborhood $V \in \mathcal{K}$ in G.

Then $V \cap G^u$ (respectively, $V \cap G_u$) is a compact neighborhood of x in G^u (respectively, in G_u).

Let us also notice that if G is a locally generalized conditionally compact groupoid and if R is the principal groupoid associated with G, then R endowed with the transported topology from G (Definition 2.1) is a Hausdorff topological groupoid. (Remark 2.4 and Proposition 2.6).

Let us justify the fact that we call a set $K \in \mathcal{K}$ generalized conditionally compact. Let G be a topological groupoid with locally compact Hausdorff unit space. Let K be an r-compact subset of G in the sense of [4], meaning that $K \cap r^{-1}(V)$ is compact for any compact subset V of $G^{(0)}$. Let $(x_i)_i$ be a net in G converging to x, and let V be a compact neighborhood $r(x)$ in $G^{(0)}$. Then there is i_0 such that $L_0 = \{r(x_i), i \geq i_0\} \cup \{r(x)\} \subset V$. Thus there is a compact set $K_0 = K \cap r^{-1}(V)$ such that $K \cap r^{-1}(L_0) \subset K_0 \subset K$.

Proposition 3.4. *Let G be a topological groupoid with Hausdorff unit space and let K be a subset of G. Let us consider the following conditions:*

- (i)**:** *For every net $(x_i)_i$ in G converging to x, there is i_0 and a compact set K_0 such that $K \cap r^{-1}(L_0) \subset K_0 \subset K$, where $L_0 = \{r(x_i),\ i \geq i_0\} \cup \{r(x)\}$.*
- (ii)**:** *For every net $(x_i)_i$ in G converging to x, there is i_0 and a compact set K_0 such that $K \cap (r,d)^{-1}(L) \subset K_0 \subset K$, where $L = \{(r,d)(x_i),\ i \geq i_0\} \cup \{(r,d)(x)\}$.*
- (iii)**:** *$(r,d)(K)$ is closed in R endowed with the quotient topology (or equivalently, $(r,d)^{-1}((r,d)(K))$ is closed in G).*

Then $(i) => (ii) => (iii)$.

Proof. $(i) => (ii)$. It easily follows from $K \cap (r,d)^{-1}(L) \subset K \cap r^{-1}(L_0)$.

$(ii) => (iii)$. Let $(x_i)_i$ be a net in $(r,d)^{-1}((r,d)(K))$ converging to x in G. Since $x_i \to x$, it follows that there is i_0 and a compact set K_0 such that $K \cap (r,d)^{-1}(L) \subset K_0 \subset K$, where $L = \{(r,d)(x_i),\ i \geq i_0\} \cup \{(r,d)(x)\}$. On the other hand, $x_i \in (r,d)^{-1}((r,d)(K))$ implies that there is $y_i \in K$ such that

$$(r(x_i), d(x_i)) = (r(y_i), d(y_i)).$$

Consequently, $y_i \in K \cap (r,d)^{-1}(L) \subset K_0$. Hence $(y_i)_i$ has a convergent subnet, also denoted $(y_i)_i$, $y_i \to z \in K_0 \subset K$. In the Hausdorff space $G^{(0)}$, we have $r(x_i) = r(y_i) \to r(z)$ and on the other hand $r(x_i) \to r(x)$. Thus $r(x) = r(z)$. Similarly, $d(x) = d(z)$ and therefore $x \in (r,d)^{-1}((r,d)(K))$. □

Example 3.5. (1) Let G be a locally compact groupoid having Hausdorff unit space. Then G endowed with the family of compact subsets satisfies the conditions of the Definition 3.2.

(2) If G is a locally conditionally compact groupoid in the sense of [4], then G endowed with the family of conditionally compact subsets (in the sense of [4]) is a locally generalized conditionally compact groupoid in the sense of Definition 3.2.

(3) We shall provide an example of a subset K of a locally compact Hausdorff groupoid G such that K is generalized conditionally compact without being conditionally compact (in fact without being neither r-compact nor d-compact). Also we shall prove that the product topology as well as the quotient topology on the principal groupoid associated with G is not locally conditionally compact (in the sense of [4]).

Let us consider the action of $\mathbb{Z}$, the group of integer numbers, on $[0,\infty)$ given by

$$u \cdot n = u^{2^n},\ u \in [0,\infty),\ n \in \mathbb{Z}$$

and let $G = [0,\infty) \times \mathbb{Z}$ (endowed with the product topology) be the corresponding r-discrete groupoid. Then G is a locally compact Hausdorff groupoid (thus, in particular G endowed with the family of compact subsets is a locally generalized conditionally compact groupoid).

Let us prove that $K = [2,\infty) \times \mathbb{Z}$ is neither r-compact, nor d-compact, but it is generalized conditionally compact. Let $L_0 = [1,3]$. Then $r^{-1}(L_0) \cap K = [2,3] \times \mathbb{Z}$ is not compact in G. Hence $K = [2,\infty) \times \mathbb{Z}$ is not r-compact. Also let us notice that

$$d^{-1}(L_0) \cap K \supset \{(t,n) :\ 2 \le t \le 3,\ n \le 0\}.$$

Thus $K = [2, \infty) \times \mathbb{Z}$ is not d-compact. Let $(x_i, n_i)_{i \in I}$ be a net in G converging to (x, n). There is i_0 such that $n_i = n$ for all $i \geq i_0$. Also if L is compact neighborhood of x in $[0, \infty)$, there is $i_1 \geq i_0$ such that $x_i \in L$ for all $i \geq i_1$. If we denote $L_1 = L \setminus \{0, 1\}$, then

$$\begin{aligned} & K \cap (r,d)^{-1}\left((r,d)\left(L \times \{n\}\right)\right) \\ \subset\ & K \cap (r,d)^{-1}\left(\{(0,0)\} \cup (r,d)\left(L_1 \times \{n\}\right) \cup \{(1,1)\}\right) \\ =\ & K \cap \left(\left(\{0\} \times \mathbb{Z}\right) \cup \left(L_1 \times \{n\}\right) \cup \left(\{1\} \times \mathbb{Z}\right)\right) \\ =\ & K \cap \left(L_1 \times \{n\}\right) \\ =\ & \left([2,\infty) \cap L_1\right) \times \{n\} \\ =\ & \left([2,\infty) \cap L\right) \times \{n\} \end{aligned}$$

is compact. Therefore $K = [2, \infty) \times \mathbb{Z}$ is generalized conditionally compact.

Let us describe the transported topology (Definition 2.1) from $G = [0, \infty) \times \mathbb{Z}$ on R, the principal groupoid associated with G. The isotropy group $G_u^u = \{0\}$ for all $u \in (0, 1) \cup (1, \infty)$ and $G_u^u = \mathbb{Z}$ for $u \in \{0, 1\}$. Since R can be written as

$$\{(0,0)\} \cup \{(1,1)\} \cup \left\{\left(u, u^{2^n}\right) : u \in (0,\infty) \setminus \{1\}\ \ , n \in \mathbb{Z}\right\},$$

and since the restriction of (r, d) to $((0, \infty) \setminus \{1\}) \times \mathbb{Z}$ is a homeomorphism, it follows that on $R \cap (((0, \infty) \setminus \{1\}) \times \mathbb{Z})$ the transported topology from G, the quotient topology and the product topology coincide. Since for $\varepsilon > 0$ and $n \neq m$,

$$(r,d)\left([0,\varepsilon) \times \{n\}\right) \cap (r,d)\left([0,\varepsilon) \times \{m\}\right) = \{(0,0)\}$$

it follows that $\{(0, 0)\}$ is open with respect to transported topology from G. Similarly $\{(1, 1)\}$ is open with respect to transported topology from G. Obviously, $\{(0, 0)\}$ and $\{(1, 1)\}$ are not open for quotient topology and consequently, they are not open for the product topology. It is not difficult to see that R endowed with the product topology (induced from $[0, \infty) \times [0, \infty)$) is not locally closed and therefore product topology on R is not locally compact (in order to see that, let us notice that $(t, 0) = \lim_{n \to \infty} \left(t, t^{2^n}\right) \in \overline{R}$ for all $t \in [0, 1)$, and since for all $n > 0$, $\left(\frac{1}{n}, 0\right) \in \overline{R} \setminus R$ and $\lim_{n \to \infty} \left(\frac{1}{n}, 0\right) = (0, 0) \in R$, it follows that R is not open in $\overline{R}$).

Since the unit space of R is a locally compact space and since the topology on R is not locally compact, it follows that the topology on R can not be locally conditionally compact.

Let us also prove that the quotient topology on R is not locally conditionally compact. Let us assume that K_0 is a conditionally compact neighborhood (in the sense of [4]) of $\{(0,0)\}$ with respect to the quotient topology. Let V be an open set such that $(0,0) \in V \subset K_0$. Since $(r,d)^{-1}(V)$ is an open set containing $\{0\} \times \mathbb{Z}$, it follows that for all n there is $\eta_n > 0$ such that $[0,\eta_n) \times \{n\} \subset (r,d)^{-1}(V)$. Let

$$\varepsilon_n = \sup\left\{\eta \leq 1 : \left(t, t^{2^n}\right) \in V \text{ for all } \ 0 \leq t \leq \eta\right\}.$$

Since for all $t > \varepsilon_n$, $\left(t, t^{2^n}\right) \notin V$ and since V is open, it follows that

$$\left(\varepsilon_n, (\varepsilon_n)^{2^n}\right) = \lim_{t \to \varepsilon_n +} (r,d)(t,n) \notin V.$$

Also since K_0 is closed and since for all $t < \varepsilon_n$, $\left(t, t^{2^n}\right) \in V$, it follows that

$$\left(\varepsilon_n, (\varepsilon_n)^{2^n}\right) = \lim_{t \to \varepsilon_n -} (r,d)(t,n) \in \overline{V} \subset K_0.$$

Let $\varepsilon = \inf\limits_{n \geq 0} \varepsilon_n$. If $\varepsilon = 0$, then there is a subsequence $\varepsilon_{n_j} \to 0$. Since $K_0 \cap r^{-1}([0,1])$ is compact, $\left(\left(\varepsilon_{n_j}, \left(\varepsilon_{n_j}\right)^{2^{n_j}}\right)\right)_j$ has a convergent subsequence with respect to the quotient topology. Since the limit (u,v) of this subsequence with respect to the quotient topology is the same as the limit with respect to the product topology, it follows that $(u,v) = (0,0)$. But we have noticed that $\left(\varepsilon_{n_j}, \left(\varepsilon_{n_j}\right)^{2^{n_j}}\right) \notin V$ and hence $(0,0) \notin V$. This is a contradiction. Thus $\inf\limits_{n \geq 0} \varepsilon_n = \varepsilon > 0$ and consequently, $\left(\frac{\varepsilon}{2}, \left(\frac{\varepsilon}{2}\right)^{2^n}\right) \in V \subset K_0$. Since $K_0 \cap r^{-1}([0,1])$ is compact, $\left(\left(\frac{\varepsilon}{2}, \left(\frac{\varepsilon}{2}\right)^{2^n}\right)\right)_n$ has a convergent subsequence whose limit point is $\left(\frac{\varepsilon}{2}, 0\right) \notin R$. This is a contradiction also.

According to the next proposition, R endowed with the transported topology from G is a locally generalized conditionally compact groupoid.

Proposition 3.6. *Let G be a locally generalized conditionally compact groupoid with respect to $\mathcal{K}$ (in the sense of Definition 3.2), and let R be the principal groupoid associated with G. Then R endowed with the transported topology from G (Definition 2.1) is a locally generalized conditionally compact compact groupoid with respect to*

$$\mathcal{R} = \left\{ \bigcap_{K\in\mathcal{F}} (r,d)(K),\ \mathcal{F} \text{ finite collection of sets } K \in \mathcal{K} \right\}.$$

Proof. According to Proposition 2.6 and Remark 2.4, R is a Hausdorff topological groupoid. By Lemma 2.7 and since $(r,d)(K) \cap R^u = (r,d)(K \cap G^u)$ it follows that $(r,d)(K) \cap R^u$ is compact for all $K \in \mathcal{K}$. Let us note that the transported topology from G to R seen as principal associated groupoid to G coincides with the transported topology from R to R seen as principal associated groupoid to itself. □

Proposition 3.7. *Let G be a locally generalized conditionally compact groupoid with respect to $\mathcal{K}$ (in the sense of Definition 3.2), let R be the principal groupoid associated with G and let $\tau_{G\vee R}$ be the modified topology on G with respect to R (Definition 2.9). Then $(G, \tau_{G\vee R})$ is a locally generalized conditionally compact compact groupoid with respect to*

$$\mathcal{K}_{G\vee R} = \left\{ \begin{array}{c} \bigcap_{K\in\mathcal{F}} (r,d)^{-1}\left((r,d)(K)\right) \cap C, \\ \mathcal{F} \text{ finite collection of sets } K \in \mathcal{K},\ C \in \mathcal{K} \end{array} \right\}.$$

Proof. It suffices to notice that $(r,d)(K)$ is closed with respect to quotient topology on R (Proposition 3.4) and therefore

$$\bigcap_{K\in\mathcal{F}} (r,d)^{-1}\left((r,d)(K)\right)$$

is closed in G, for every $\mathcal{F}$ finite collection of sets $K \in \mathcal{K}$. □

Proposition 3.8. *Let (G, τ_G) be a second countable topological groupoid which is locally generalized conditionally compact groupoid with respect to a family of generalized conditionally compact subsets $\mathcal{K}$ (in the sense of Definition 3.2), and let R be the principal groupoid associated with G. Let $\tau_{G\vee R}$ be the modified topology on G with respect to R (Definition 2.9). Then*

(1) *The transported topology on* R *from* G *(Definition 2.1) and the quotient topology on* R *generate the same Borel structure.*
(2) *The Borel structures on* (G, τ_G) *and* $(G,\ \tau_{G\vee R})$ *are the same (the Borel sets of a topological space are taken to be the* σ*-algebra generated by the open sets).*

Proof. 1. The transported topology from G on R is finer than the quotient topology on R. Therefore it suffices to prove that for every $U \in \tau_G$, $(r,d)(U)$ is Borel with respect to the Borel structure generated by the quotient topology on R. Since τ_G is second countable and each point $x \in G$ has a local base of neighborhoods belonging to $\mathcal{K}$, it follows that U can be represented as a countable union of $K_n \in \mathcal{K}$. Therefore

$$(r,d)(U) = \bigcup_n (r,d)(K_n)$$

is Borel (by Proposition 3.4 $(r,d)(K_n)$ is closed with respect to quotient topology of R).

2. It follows from the fact that $\tau_{G\vee R}$ is finer than τ_G and $(r,d):(G,\ \tau_G) \to R$ is continuous when R is endowed with the quotient topology. □

Proposition 3.9. *Let* G *be a locally generalized conditionally compact groupoid with respect to* $\mathcal{K}$ *(in the sense of Definition 3.2), let* τ_G *be its topology,* $\tau_{G^{(0)}}$ *the topology induced on* $G^{(0)}$ *and let* R *be the principal groupoid associated with* G*. Suppose that* R *is endowed with* $\tau_R(\tau_G)$ *the transported topology from* G *(Definition 2.1) and let* $\tau_{G\vee R}$ *be the modified topology on* G *with respect to* R *(Definition 2.9).*

(1) *If* S *is a Hausdorff subspace of* G *and* F *is a closed subset of* G *with respect to* τ_G *(respectively,* $\tau_{G\vee R}$*) such that* $F \subset S$*, then* $F \cap K$ *is closed with respect to* τ_G *(respectively,* $\tau_{G\vee R}$*) for every* $K \in \mathcal{K}$*.*
(2) *If* S *is a Hausdorff subspace of* G *and* $K \subset S$*,* $K \in \mathcal{K}$*, then* K *is closed in* S *with respect to* τ_G*.*

Proof. 1. Let $(x_i)_i$ be a net in $K \cap F$ such that $x_i \to x$. Condition 5 from Definition 3.1 implies that there is i_0 and a compact set K_0 such that $K \cap (r,d)^{-1}(L) \subset K_0 \subset K$ where $L = \{(r,d)(x_i),\ i \geq i_0\} \cup \{(r,d)(x)\}$. Hence $(x_i)_i$ has a convergent subnet to an element $y \in K$ with respect to τ_G. Since $G^{(0)}$ is Hausdorff, $(r(x), d(x)) = (r(y), d(y))$, and consequently, if $x_i \to x$ with respect to $\tau_{G \vee R}$, then $x_i \to y$ with respect to $\tau_{G \vee R}$. Since F is closed, $y \in F \subset S$. But S is Hausdorff, and therefore $x = y \in K \cap F$.

2. Let $(x_i)_i$ be a net in K such that $x_i \to x$ in S. Condition 5 from Definition 3.1 implies that there is i_0 and a compact set K_0 such that $K \cap (r, d)^{-1}(L) \subset K_0 \subset K$ where $L = \{(r, d)(x_i), \ i \geq i_0\} \cup \{(r, d)(x)\}$. Hence $(x_i)_i$ has a convergent subnet to an element $y \in K \subset S$ with respect to τ_G. Since S is Hausdorff, $x = y \in K$. □

Definition 3.10. Let G be a locally generalized conditionally compact groupoid with respect to $\mathcal{K}$ (in the sense of Definition 3.2). If V is an open subset of G and $K \in \mathcal{K} \cup \{\emptyset\}$ then $V \setminus K$ form a basis for a topology on G. Let us denote this topology by $\tau_{\mathcal{K}}$ and call it the *topology induced by* $\mathcal{K}$.

If V is an open subset of G, $\mathcal{F}$ is a finite collection of open subsets of G and $K \in \mathcal{K} \cup \{\emptyset\}$ then

$$V \bigcap \left(\bigcap_{U \in \mathcal{F}} (r, d)^{-1} ((r, d)(U)) \right) \setminus K$$

form a basis for a topology on G. Let us denote this topology by $\tau_{\mathcal{K} \vee R}$ and call it the *modified topology on G with respect to R induced by $\mathcal{K}$.*

Proposition 3.11. *If G is a locally generalized conditionally compact groupoid with respect to $\mathcal{K}$ (in the sense of Definition 3.2), then G endowed with $\tau_{\mathcal{K}}$ (and hence $\tau_{\mathcal{K} \vee R}$) is a Hausdorff space.*

Proof. Let $x_1 \neq x_2$ be two points in G. If $r(x_1) \neq r(x_2)$, then there is an open neighborhood $U \subset G^{(0)}$ of $r(x_1)$ and an open neighborhood $V \subset G^{(0)}$ of $r(x_2)$ such that $U \cap V = \emptyset$ (because $G^{(0)}$ is Hausdorff). Then $r^{-1}(U)$ and $r^{-1}(V)$ are disjoint open neighborhoods of x_1, respectively x_2. Similarly, if $d(x_1) \neq d(x_2)$, then there are two disjoint neighborhoods of x_1 and x_2. If $r(x_1) = r(x_2)$ and $d(x_1) = d(x_2)$, then x_1, x_2 are in the Hausdorff space G^u where $u = r(x_1)$. Since each $x \in G$ has a fundamental system of neighborhoods belonging to $\mathcal{K}$, it follows that there are K_1, $K_2 \in \mathcal{K}$ neighborhoods of x_1, respectively x_2 such that $K_1 \cap K_2 \cap G^u = \emptyset$.

If $K_1 \cap K_2 = \emptyset$ the proof is complete. If $K_1 \cap K_2 \neq \emptyset$, then $K_1 \setminus (K_1 \cap K_2)$ and $K_2 \setminus (K_1 \cap K_2)$ are disjoint open neighborhoods of x_1, respectively x_2 (with respect to $\tau_{\mathcal{K}}$). □

Let (G, τ_G) be a second countable locally Hausdorff topological groupoid which is locally generalized conditionally compact groupoid with respect to a family of generalized conditionally compact subsets $\mathcal{K}$ (in the sense of Definition 3.2), and let R be the principal groupoid associated with G. Let $\tau_{G\vee R}$ be the modified topology on G with respect to R (Definition 2.9), $\tau_{\mathcal{K}}$ and $\tau_{\mathcal{K}\vee R}$, the topologies induced by $\mathcal{K}$ (Definition 3.10).

It is easy to see that:

(1) A net $(x_i)_i$ converges to x with respect to $\tau_{\mathcal{K}}$ if and only if $(x_i)_i$ converges to x with respect to τ_G and for every $K \in \mathcal{K}$ with $x \notin K$, there is i_K such that $x_i \notin K$ for all $i \geq i_K$.

(2) A net $(x_i)_i$ converges to x with respect to $\tau_{\mathcal{K}\vee R}$ if and only if $(x_i)_i$ converges to x with respect to τ_G, $(r(x_i), d(x_i))_i$ converges to $(r(x), d(x))$ in R with respect to the transported topology from G (Definition 2.1) and for every $K \in \mathcal{K}$ with $x \notin K$, there is i_K such that $x_i \notin K$ for all $i \geq i_K$.

(3) $\tau_{\mathcal{K}}$, τ_G induce the same subspace topology on every closed Hausdorff subspace S of G.

(4) $\tau_{\mathcal{K}\vee R}$, $\tau_{G\vee R}$ induce the same subspace topology on every closed Hausdorff subspace S of G.

(5) For every $u \in G^{(0)}$, $\tau_{\mathcal{K}\vee R}$, $\tau_{G\vee R}$, $\tau_{\mathcal{K}}$ and τ_G induce the same subspace topology on G^u.

(6) If $U \in \tau_G$ is a Hausdorff set, $K \in \mathcal{K}$, $K \subset U$, then every function $f : G \to \mathbb{C}$ which is τ_G-continuous on U and vanishes outside K, is $\tau_{\mathcal{K}}$-continuous on G.

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