A note on n-continuous L*-operators

by

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Abstract. We introduce the concept of an $n$–continuous $L^*$–operator for $n = 1, 2, ...$. The class of spaces that admit $n$–continuous $L^*$–operators includes as diverse objects as topological vector $F$–spaces and compact finite dendrites. We prove some fixed point theorems within the framework of the class of $L^*$–spaces. We show an example of a non-continuous $L^*$–operator which is $n$–continuous for each $n = 1, 2, ...$.

For a topological space $X$, let $Fin(X)$ and $\exp(X)$ denote, respectively, the set of all finite non-empty subsets of $X$, and the set of all non-empty subsets of $X$. Following [6], an $L^*$–operator on $X$ is a function $\Lambda: Fin(X) \to \exp(X)$ that satisfies the following condition:

(*) If $A \in Fin(X)$ and $\{U_x: x \in A\}$ is an open cover of $X$, then there exists $B \in Fin(X)$ such that $B \subseteq A$ and $\Lambda(B) \cap \bigcap \{U_x: x \in B\} \neq \emptyset$.

A topological space $X$ with an $L^*$–operator $\Lambda$ on it is referred to as an $L^*$–space and it is denoted by $(X, \Lambda)$. For an $L^*$–space $(X, \Lambda)$, a set $Y \subseteq X$ is said to be $L^*$–convex if $\Lambda(A) \subseteq Y$ for each $A \in Fin(Y)$. The family $CON(X, \Lambda)$ of all $L^*$–convex subsets of $X$ constitutes a convexity structure on $X$.

Examples of $L^*$–operators, and thus of $L^*$–spaces, abound. In fact, one can define an $L^*$–operator on arbitrary topological space $X$: set $\Lambda(A)$ to be an any dense subset of $X$. Taking the convex hull of a finite set provides another example of an $L^*$–operator. This is not an obvious fact and its proof is based on the following theorem due to the first author [4]. (In the sequel, $\Delta_n \subseteq \mathbb{R}^{n+1}$ denotes the unit simplex in $\mathbb{R}^{n+1}$ and $\Delta_J$ its face.)

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Theorem 1. Let $\sigma : \Delta_n \to X$ be a continuous function. If $U_1, U_2, \ldots, U_{n+1}$ are open subsets of $X$ that cover $\sigma(\Delta_n)$, then there exists $J \subseteq \{1, 2, \ldots, n+1\}$ such that $\sigma(\Delta_J) \cap \bigcap \{U_i : i \in J\} \neq \emptyset$.

Proposition 1. If $X$ is a topological vector space, then, $\Lambda(A) = \text{con}(A)$, defines an $L^*$-operator on $X$ such that $\text{CON}(X, \Lambda)$ is identical with the class of convex subsets of $X$.

Proof. Let $A = \{a_1, a_2, \ldots, a_{n+1}\} \in \text{Fin}(X)$. Take the unit $n$-dimensional simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$ and consider the affine map $\sigma : \Delta_n \to X$ that carries the set of the vertices of $\Delta_n$ onto $A$. Under such circumstances, $\sigma(\Delta_n) = \text{con}(A)$ and Theorem 1 applies. The second part of the proposition follows immediately as well.

Let us point out that the convexity structures induced by $L^*$-operators encompass properly $L$-structures introduced by Park and Ben-El-Mechaiekh’s et al. [1]. A more detailed exposition on $L$-spaces and related topics is presented in [5].

Let $X$ be a topological vector space. Recall that a non-negative function $|||\cdot|||$ is called an $F$-norm (or seminorm) on $X$ if $||x + y|| \leq ||x|| + ||y||$ and $||tx|| \leq ||x||$ for all $x, y \in X$, $0 \leq t \leq 1$; and the topology of $X$ is generated by the open balls $B(p, r) = \{x : ||x - p|| < r\}$.

Proposition 2. Let $X$ be a convex subspace of topological vector space endowed with an $F$-norm $|||\cdot|||$. Then the $L^*$-operator on $X$ given by $\Lambda(A) = \text{con}(A)$ is $n$-continuous for each $n$.

Proof. Since $||x - y||$ is translation invariant, the $n$-continuity of $\Lambda$ will follow from the $n$-continuity of $\Lambda$ at $0$. To show that, we are going to estimate the diameter of the convex hull of a finite set. This issue is a subject of serious research (cf. [3]). Luckily, we only have to provide a rough estimate, which can be easily accomplished in the following manner.
Let $A = \{a_1, a_2, \ldots, a_n\} \subseteq X$ and let $x, y \in \text{con} (A)$. If $y = \sum_{i=1}^{n} t_i a_i$, where $\sum_{i=1}^{n} t_i = 1$ and each $t_i \geq 0$, then $\|x - y\| = \left\|\sum_{i=1}^{n} t_i x - \sum_{i=1}^{n} t_i a_i\right\| = \left\|\sum_{i=1}^{n} (t_i x - t_i a_i)\right\| \leq \sum_{i=1}^{n} \|x - a_i\|$. Redoing the same calculations for each $\|x - a_i\|$, we get:

$\|x - y\| \leq \sum_{i=1}^{n} \|x - a_i\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|a_j - a_i\| \leq n^2 \cdot \text{diam}A.$

What ensues is that if $U$ is the ball at $0$ of radius $r > 0$, then $\Lambda(A) = \text{con} (A) \subseteq U$ provided that $A$ is contained in the ball at $0$ of radius less than $\frac{r}{2n^2}$ and $|A| \leq n$. Thus $\Lambda$ is $n$–continuous.

Let $X \neq \emptyset$ be a normal topological space and let $n$ be a natural number or $0$. The space $X$ is said to have the covering dimension $\leq n$ if every open cover of the space $X$ has a finite open refinement of order $\leq n$ (i.e., each point of $X$ belongs to at most $n + 1$ elements of the refinement).

**Theorem 2.** Let $(X, \Lambda)$ be an $L^*$-space, where $X$ is a Hausdorff space, and let $g : X \to X$ be a continuous function such that the closure of $g(X)$, $\overline{g(X)}$, is a compact subspace of $X$. If $\overline{g(X)}$ has the covering dimension $\leq n - 1$ and $\Lambda$ is $n$–continuous at each point of $\overline{g(X)}$, then $g$ has a fixed point.

**Proof.** Suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. Since $X$ is a Hausdorff space, for each $x \in X$ there exists an open neighborhood $W_x$ of $x$ such that $W_x \cap g(W_x) = \emptyset$. For each $x \in g(X)$, pick an open neighborhood $V_x$ of $x$ that is contained in $W_x$ and verifying $\Lambda(A) \subseteq W_x$ provided $A \subseteq V_x$ and $|A| \leq n$. Since $g(X)$ is a compact Hausdorff space, there exists a relatively open finite covering $U = \{U_1, \ldots, U_m\}$ of $g(X)$ which is a barycentric refinement of the family $\{V_x : x \in g(X)\}$ (cf. Engelking [2], Theorem 5.1.12), i.e., for each $y \in g(X)$ there exists $x \in g(X)$ such that $\text{st} (y, U) = \bigcup \{U \in U : y \in U\} \subseteq V_x$. Since the covering dimension of $\overline{g(X)}$ is at most $n - 1$, we may also assume that $U$ is of order $\leq n - 1$, i.e., if $J \subseteq \{1, 2, \ldots, m\}$ and $\bigcap \{U_i : i \in J\} \neq \emptyset$, then $|J| \leq n$. Choose points $x_i \in U_i \cap g(X)$ and set $A = \{x_1, \ldots, x_m\}$. Since $\{g^{-1}(U_i) : i = 1, \ldots, m\}$ is an
open cover of $X$, there exist $1 \leq i_1 < \cdots < i_k \leq m$ and a point $w \in X$ such that $w \in \Lambda(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}) \cap g^{-1}(U_{i_1}) \cap \cdots \cap g^{-1}(U_{i_k})$. Consequently, $k \leq n$. Since $g(w) \in U_{i_1} \cap \cdots \cap U_{i_k}$ and since $x_i \in U_i$, \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subseteq U_{i_1} \cup \cdots \cup U_{i_k} \subseteq \text{st}(g(w), U) \subseteq V_x$ for some $x \in g(X)$. Since $|\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}| \leq n$, $w \in \Lambda(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}) \subseteq W_x$ and therefore $g(w) \notin W_x$. From the other hand side, $g(w) \in \text{st}(g(w), U) \subseteq V_x \subseteq W_x$, which is a contradiction. □

Proposition 2 and Theorem 2 yield the following corollaries.

**Corollary 1.** Let $X$ be homeomorphic to a convex subset of a topological vector space endowed with an $F$-operator $\|\|$. If $g : X \to X$ is continuous and $g(X)$ is compact and has a finite covering dimension, then $g$ has a fixed point.

**Corollary 2.** If $X$ is homeomorphic to the unit circle, then $X$ does not admit an $L^*$-operator that is $2$-continuous.

An $L^*$-operator $\Lambda$ is called continuous at a point $p \in X$ if each open neighborhood $U$ of $p$ contains a neighborhood $V$ of $p$ verifying that $\Lambda(A) \subseteq U$ provided that $A \in \text{Fin}(V)$. We say that $\Lambda$ is continuous on a set $Y$ if $\Lambda$ is continuous at each point of $Y$. The class of spaces that admit continuous $L^*$-operators contains locally convex topological vector spaces, connected linearly ordered topological spaces, and compact finite dendrytes among others. For a more detailed treatment of this subject, see [5].

By making only minor changes in the proof of Theorem 2, we get the following fixed point theorem.

**Theorem 3.** Let $(X, \Lambda)$ be an $L^*$-space, where $X$ is a Hausdorff space and let $g : X \to X$ be a continuous function such that $g(X)$ is a compact subspace of $X$. If the operator $\Lambda$ is continuous at each point of $g(X)$, then $g$ has a fixed point.

Clearly, any continuous $L^*$-operator is $n$-continuous for each $n = 1, 2, 3, \ldots$. The following example shows that the converse may not hold true.

**Example 1.** Let $X = L_p([0,1])$ be the topological vector space of all Lebesgue integrable real functions on the interval $[0,1]$ endowed with the $F$-norm given by $\|f\| = \int_0^1 |f(x)|^p \, dx$. According to
Proposition 2, \( \Lambda(A) = \text{con}(A), A \in \text{Fin}(X) \), is an \( L^*- \) operator on \( X \) which is \( n \)-continuous for each \( n \). For any \( Y \subseteq X \), the set \( \Lambda(Y) = \bigcup \{ A \in \text{Fin}(Y) \} = \bigcup \{ \text{con}(A) : A \in \text{Fin}(Y) \} \) is convex and contains \( Y \). Indeed, if \( x, y \in \Lambda(Y) \), say \( x \in \text{con}(A) \), \( y \in \text{con}(B) \), then the segment \([x, y] \subseteq \text{con}(A \cup B) \subseteq \Lambda(Y) \). However, if \( 0 < p < 1 \), then \( X \) is the only non-empty convex subset of \( X \) with a non-empty interior (see Rudin [7]). Hence the operator \( \Lambda \) cannot be continuous if \( 0 < p < 1 \).

References