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INVERSE LIMITS AND JULIA SETS

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Abstract. We investigate inverse limits of quadratic polynomial maps on Julia Sets, and show that they are closely related to the inverse limits of the corresponding Hubbard Tree maps. In the case where the critical point is strictly preperiodic, we show that every indecomposable continuum contained in the inverse limit of the Julia Set is in fact contained in the inverse limit of the Hubbard Tree. Similar results are shown for the case in which there is an attracting periodic point which is not an n-tupling.

1. Introduction

The inverse limit has proved to be a very useful tool for studying the dynamical system given by a continuous function $f : X \to X$ on a topological space X, as well as giving a construction of topological spaces which are interesting in their own right. If $\hat{X}$ is the inverse limit of X with respect to the single bonding map f, then there is a natural homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ which is closely related to f. For example, there is a natural one-to-one correspondence between the periodic orbits of f and the periodic orbits of $\hat{f}$. Even if f is a relatively simple map on the unit interval $[0, 1]$, the inverse limit of f can be very complicated. Nevertheless, there are many such maps where the structure of the inverse limit is fairly well understood.

Quadratic Julia Sets, known by many because of the beautiful associated computer images, are defined from quadratic functions $f : \mathbb{C} \to \mathbb{C}$. If J is the Julia Set of such a quadratic f, then $f(J) = J$, so that $f|J$ becomes a natural candidate for an inverse limit construction. In this paper, we investigate the extent to which the inverse limits of some locally connected quadratic Julia Sets can be understood based on the inverse limits of the corresponding Hubbard Trees.

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If $f : X \to X$ and Y is a subspace of X such that $f(Y) \subseteq Y$, then it is a simple consequence of the definition that the inverse limit of $f|Y$ is a subspace of the inverse limit of f. If the Julia Set J of $f(z) = z^2 + c$ is a dendrite and the critical point 0 has a finite orbit, then the smallest continuum containing the orbit of 0 is a tree T (called the *Hubbard Tree* of J), with $f(T) = T$, so that $\hat{T}$ (the inverse limit of T with respect to $f|T$) is a subspace of $\hat{J}$ (the inverse limit of J with respect to $f|J$). In this case we will show that $\hat{T}$ is the smallest subcontinuum of $\hat{J}$ which contains all of the indecomposable subcontinua of $\hat{J}$, and in many cases that $\hat{T}$ is the only indecomposable subcontinuum of $\hat{J}$.

If $f(z) = z^2 + c$ has an attracting periodic orbit, then the situation is more complicated. In this case, the Julia Set J contains circles (where by *circles* we mean homeomorphic copies of the unit circle, and not necessarily geometric circles). In the simplest case, c is located in the interior of one of the small "cardioids" of the Mandelbrot Set, and a quotient space D with quotient map $q : J \to D$ can be formed by shrinking all circles of J to points. D will be a dendrite, with $g : D \to D$ induced from f by q, and D will contain a tree T which is a Hubbard Tree. J will contain a unique circle C such that $f|C$ is not one-to-one, and $K = q^{-1}(T)$ will be a continuum. Connections between the inverse limits $\hat{J}$, $\hat{K}$, $\hat{D}$, and $\hat{T}$ will then be exploited to learn about the structure of $\hat{J}$, proving, for example, that in some cases $\hat{K}$ is the smallest subcontinuum of $\hat{J}$ which contains all indecomposable subcontinua of $\hat{J}$.

We begin with some basic notation. Let ω, $\mathbb{Z}$, $\mathbb{Z}_-$, $\mathbb{R}$, $\mathbb{C}$ be the sets of nonnegative integers, integers, negative integers, real numbers and complex numbers respectively. If $f : X \to X$ and n is a positive integer, f^n is f composed with itself n times, $f^{-n} = (f^n)^{-1}$ (as a set function if f is not one-to-one), and f^0 is the identity map. A point $x \in X$ is *periodic* with period $p \geq 1$ with respect to the function f if $f^p(x) = x$ but $f^n(x) \neq x$ for $1 \leq n \leq p-1$. A *continuum* is a compact connected metric space. An *arc* is a space homeomorphic to the closed interval $[0, 1]$. A *ray* is a one-to-one continuous image $R = f([0, 1))$ of the half-open interval $[0, 1)$, with $f(0)$ being called the *initial point* of R. A *two-sided ray* is a one-to-one continuous image of $\mathbb{R}$. A *tree* is a union of finitely many arcs which contains no circles. A *dendrite* is a locally connected, uniquely arcwise connected continuum. A continuum is *decomposable* if it can be written as the union of two proper subcontinua. A continuum which is not decomposable is *indecomposable*.

If X is a topological space and $f : X \to X$, then the *inverse limit* of X with respect to f, written (X, f) or more often $\hat{X}$, is the set of all doubly infinite sequences $\hat{x} = \langle x_n : n \in \mathbb{Z} \rangle$ from X such that $x_{n+1} = f(x_n)$ for all n, with the subspace topology induced from the product space $X^{\mathbb{Z}}$.

Note that in this definition of the inverse limit, the interesting information appears in the negative indices, because the map taking $\hat{x}$ to $\hat{x}|\mathbb{Z}_-$ gives a natural bijection between $\hat{X}$ and $\{\hat{x}|\mathbb{Z}_- : \hat{x} \in \hat{X}\}$, and the inverse limit is often indexed thus on the negative integers, or in the opposite order on the nonnegative integers. Allowing all integers to be used gives an equivalent formulation and shortens many of the arguments slightly. The *shift map* $\hat{f}:\hat{X}\to\hat{X}$ is defined by $\hat{f}(\langle x_n : n\in\mathbb{Z}\rangle)=\langle f(x_n) : n\in\mathbb{Z}\rangle = \langle x_{n+1} : n\in\mathbb{Z}\rangle$. For $n\in\mathbb{Z}$, let $\pi_n(\hat{x}) = x_n$. If $A_n \subseteq X_n$ such that $f(A_n) = A_{n+1}$, the inverse limit of the A_n's is defined to be $\hat{A}=\{\hat{x}\in\hat{X} : x_n\in A_n \text{ for all } n\}$. The following are some of the basic well known facts about inverse limits.

Proposition 1.1. *Let X, f, $\hat{X}$, $\hat{f}$, π_n be as above. Then*

(1) *The shift map $\hat{f}$ is a homeomorphism of $\hat{X}$.*
(2) *If X is a continuum, then $\hat{X}$ is a continuum.*
(3) $\pi_n \circ \hat{f} = f\circ\pi_n = \pi_{n+1}$.
(4) *If X is a compact metric space, $\hat{A}$ is a closed subspace of $\hat{X}$, and $A_n = \pi_n(\hat{A})$ for all n, then $A_{n+1} = f(A_n)$ for all n, the inverse limit of the A_n's is exactly $\hat{A}$, and $\hat{A}$ is connected if and only if every A_n is connected. [Note that this does not work in general for subsets which are not closed.]*

Proof. (Outline) For (1), the shift map is obviously continuous, and the shift in the other direction is its continuous inverse. (3) is an immediate consequence of the fact that if $\hat{x}\in\hat{X}$, then $f(x_n) = x_{n+1}$. Since (2) follows from (4), we only need to prove the latter. $A_{n+1} = f(A_n)$ follows immediately from (3). If $\hat{x}\in\hat{A}$, then $x_n\in A_n$, so $\hat{x}$ is in the inverse limit of the A_n's. If $\hat{x}$ is in the inverse limit of the A_n's then for each n pick $\hat{x}^n \in \pi_n^{-1}(x_n)$. Then $\hat{x}^n\in\hat{A}$, and $\hat{x}$ is in the closure of $\{\hat{x}^n : n\in\omega\}$ and therefore in $\hat{A}$. If $f^{-1}(x)$ has at least two elements for all $x\in X$, then let $\hat{A}$ be $\hat{X}$ minus a point, and we have an example that the word "closed" cannot be removed. Finally, if $\hat{A}$ is not connected, then it can be written as the union of two disjoint clopen subsets $\hat{B}$ and $\hat{C}$. Then $f(B_n\cap C_n)\subseteq B_{n+1}\cap C_{n+1}$, and the sets $B_n\cap C_n$ cannot all be nonempty, since they would then have a nonempty inverse limit, contradicting that $\hat{B}$ and $\hat{C}$ don't intersect. Thus $B_n\cap C_n$ is empty for some $n\in\omega$, giving a separation of A_n. □

Definition 1.2. If $f: X\to X$ is continuous, then f is *locally one-to-one* at a point $x\in X$ iff there is a neighborhood of x on which f is one-to-one. A point $t\in X$ is a *turning point* of f if f is not locally one-to-one at t (generally only applied when X is a dendrite).

Definition 1.3. A continuous function $f : X \to X$ on a topological space X is called *locally eventually onto* (abbreviated "l.e.o.") iff for every nonempty open set U there is a positive integer n such that $f^n(U) = X$.

The following fact is well known.

Theorem 1.4. *If $f : T \to T$ is a l.e.o. tree map, then the inverse limit $\hat{T}$ of T with respect to f is indecomposable.*

Proof. (Outline) Since f is l.e.o., there is a positive integer k such that if X and Y are subtrees of T with $X \cup Y = T$, then at least one of $f^k(X)$ and $f^k(Y)$ is all of T. If $\hat{X}$ and $\hat{Y}$ are subcontinua of $\hat{T}$ such that $\hat{X} \cup \hat{Y} = \hat{T}$, then let $X_n = \pi_n(\hat{X})$ and $Y_n = \pi_n(\hat{Y})$. Then $X_n \cup Y_n = T$, and therefore at least one of $X_{n+k} = f^k(X_n)$ and $Y_{n+k} = f^k(Y_n)$ is all of T. Thus, we must either have $X_n = T$ for all n or $Y_n = T$ for all n, so that either $\hat{X}$ or $\hat{Y}$ is all of $\hat{T}$. □

Definition 1.5. Let $f : X \to X$, and let $\mathcal{S} = \{S_a : a \in \Sigma\}$ be a partition of X (i.e., a collection of pairwise disjoint sets whose union is X). The *itinerary* of a point $x \in X$ with respect to the function f and partition $\mathcal{S}$, denoted $\iota_f^{\mathcal{S}}(x)$ (or just $\iota(x)$ if $\mathcal{S}$ and f are obvious from context), is the sequence $\iota(x) = \langle a_n : n \in \omega\rangle$ given by $a_n = a$ iff $f^n(x) \in S_a$. The function f is said to have the *unique itinerary property* if distinct points have distinct itineraries.

Definition 1.6. A continuous tree map $f : T \to T$ is called *Markov* if there is a finite set $P \subseteq T$ containing all endpoints and branching points such that $f(P) \subseteq P$ and f is one-to-one on all components of $T \setminus P$. Such a P is called a *Markov partition* of T with respect to f, and the closure of a component of $T \setminus P$ is called a *P-basic interval*.

Proposition 1.7. *Let D be a dendrite, with D_0, D_1 subdendrites such that $D = D_0 \cup D_1$, and let $f : D \to D$ be a continuous function with unique turning point t such that f is one-to-one on D_0 and D_1, and suppose that f has the unique itinerary property with respect to the partition $\{\{t\}, D_0 \setminus \{t\}, D_1 \setminus \{t\}\}$. If A is any arc in J, then there is an N so that for all $n \geq N$, $f^n(A)$ contains at least three members of the orbit of t.*

Proof. If $U \subseteq D$ is a connected set with more than one point, there has to be an n such that $t \in f^n(U)$, for otherwise all elements of U would have the same itinerary. Let $A = [a, b]$, and let $n \geq 0$ be such that $t \in f^n(a, b)$, with n the least such element of ω. Then $f^n|[a, b]$ is one-to-one, and therefore neither $f^n(a)$ nor $f^n(b)$ is t. Applying the same argument to $[f^n(a), t]$, we get an $m \geq 0$ such that $t \in (f^{n+m}(a), f^m(t))$. Applying the same argument to $[t, f^m(t)]$ gives a $k \geq 0$ such that $t \in (f^k(t), f^{m+k}(t))$, so $t, f^k(t), f^{m+k}(t) \in f^{n+m+k}(A)$. Then it is easy to see that $N = n + m + k$ suffices. □

Lemma 1.8. *Let D be a dendrite, with D_0, D_1 subdendrites such that $D = D_0 \cup D_1$, and let $f : D \to D$ be a continuous function with unique turning point t such that f is one-to-one on D_0 and D_1, and suppose that f has the unique itinerary property with respect to the partition $\{\{t\}, D_0 \setminus \{t\}, D_1 \setminus \{t\}\}$. Suppose that $[\mathrm{Orb}_f(t)]$ is a tree T with e endpoints. Then $f(t), f^2(t), \cdots, f^e(t)$ are the endpoints of T.*

Proof. Note that if we ever have $f^{n_1}(t) \in \{f(t), f^2(t), \cdots, f^n(t)\}$, then $[\{f(t), f^2(t), \cdots, f^n(t)\}] = T$. Thus, supposing the lemma to be false, we must have $f^k(t) \in [t, f^{k+n}(t)]$ for some $k, n \geq 1$. Suppose that $k + n$ is the least element of ω such that this happens. Then since f is one-to-one on $[t, f^{k+n}(t)]$, $f^{k+1}(t) \in [f(t), f^{k+1+n}(t)]$, and therefore $f^{k+1}(t) \in [t, f^{k+1+n}(t)]$. Continuing by induction, we get that $f^j(t) \in [t, f^{j+n}(t)]$ for all $j \geq k$, so that $f^k(t)$ and $f^{k+n}(t)$ have the same itinerary, a contradiction. □

Lemma 1.9. *Suppose $f : T \to T$ is a Markov map with Markov set P. Suppose also that f has the unique itinerary property with respect to the partition formed by the singletons from P and the components of $T \setminus P$. Then there is a subtree W (a union of P-basic intervals) and a positive integer n such that $f^n|W$ is a l.e.o. map from W onto W. If I is any P-basic interval, there is a positive integer k such that $f^k(I)$ contains such a W.*

Proof. For each P-basic interval I, consider all trees of the form $f^n(I)$ ($n \geq 1$) such that $I \subseteq f^n(I)$. Note that if $I \subseteq f^n(I)$ and $I \subseteq f^m(I)$, then $f^{n+m}(I)$ contains both $f^n(I)$ and $f^m(I)$, so if there is any $n \geq 1$ such that $I \subseteq f^n(I)$, then there is a largest possible tree $T(I) = f^{n(I)}(I)$ containing I. If no $f^n(I)$ contains I for $n \geq 1$, let $T(I) = \emptyset$ and $n(I) = 0$. Note that if $T(I)$ is nonempty and m is any multiple of $n(I)$, then $f^m(I) = T(I)$. Thus, if J is another P-basic interval with $T(J)$ nonempty and $J \subseteq T(I)$, then $T(J) \subseteq f^{n(I)n(J)}(I) = T(I)$. Thus, there is a P-basic interval I such that $T(I)$ is nonempty and minimal. Fix such I. It is routine to show in this case that if J is a P-basic interval such that $J \subseteq T(I)$, then $T(J) = T(I)$, for otherwise minimality of $T(I)$ would be violated. If m is a multiple of all the $n(J)$'s such that $T(J) = T(I)$, then f^m maps $T(I)$ onto $T(I)$. If A is a non-trivial interval in $T(I)$, then the same argument as Proposition 1.7 shows that $f^{im}(A)$ contains a P-basic interval for some i. In that case $f^{(i+1)m}(A) = T(I)$. Thus f^m is a l.e.o. map on $T(I)$, which is our desired tree W. If I is any P-basic interval, then $T(I)$ contains a minimal $T(J)$. □

Theorem 1.10. *Let $f : T \to T$ be a tree map with subtrees T_0 and T_1 such that $T_0 \cup T_1 = T$, f is one-to-one on T_0 and T_1, and $T_0 \cap T_1 = \{t\}$, t a turning point. Suppose that f has the unique itinerary property with respect to the partition $\{\{t\}, T_0 \setminus \{t\}, T_1 \setminus \{t\}\}$. Then for every endpoint x of T, there is a subtree W containing x and a positive integer n such that $f^n|W$ is a l.e.o map from W onto W.*

Proof. We look at the proof of the previous lemma in a bit more detail. Let P be a Markov set for T, and let I be the P-basic interval containing x. Let J be a P-basic interval contained in $T(I)$. Then $t \in f^i(J)$ for some i and therefore, since x is an endpoint, $x = f^e(t) \in f^{i+e}(J)$, and therefore $I \subseteq f^{i+e}(J)$. Thus $I \cup J \subseteq f^{i+e+n(I)}(J)$, and therefore $I \subseteq T(J)$. Thus $T(I) = T(J)$ and $T(I)$ is minimal. Thus, by the argument of the previous lemma, there is an n such that $f^n|T(I)$ is a l.e.o. map from $T(I)$ onto $T(I)$. □

Corollary 1.11. *Let $f : T \to T$ be as in the previous theorem, and let $\hat{T} = (T, f)$. Then $\hat{T}$ contains an indecomposable continuum, and no proper subcontinuum of $\hat{T}$ contains all indecomposable subcontinua of $\hat{T}$.*

Proof. Let $\hat{C}$ be a proper subcontinuum of $\hat{T}$, and for each $n \in \mathbb{Z}$, let $C_n = \pi_n(\hat{C})$. Then there is an integer k such that C_n is not all of T, so there is an endpoint x of T which is not in C_k. Let W be a subtree of T containing x and n a positive integer such that $f^n|W$ is a l.e.o. map from W onto W. Define W_i by letting $W_i = W$ if $i - k$ is a multiple of n, extending the definition to other i by letting $W_{i+1} = f(W_i)$. Then the inverse limit of the W_i's is an indecomposable subcontinuum of $\hat{T}$ which is not contained in $\hat{C}$. □

2. The Dendrite Case

Let $f_c : \mathbb{C} \to \mathbb{C}$ be defined by $f_c(z) = z^2 + c$ for some fixed parameter $c \in \mathbb{C}$. The set U_c of all points z such that the sequence $\langle z, f_c(z), f_c^2(z), f_c^3(z), ...\rangle$ is unbounded is called the *basin of attraction of infinity*, the complement K_c of U_c is called the *filled-in Julia Set* of f_c, and the common boundary J_c of U_c and K_c is called the *Julia Set* of f_c. The *Mandelbrot Set* is the set $\{c \in \mathbb{C} : 0 \in K_c\} = \{c \in \mathbb{C} : J_c$ is connected$\}$. For more on the basics of Julia Sets, see [K] and the sources cited therein.

Suppose $f(z) = f_c(z)$ and that the Julia Set $J = J_c$ is a dendrite. Then $0 \in J$, and there is a ray R going to ∞ with initial point 0 but otherwise disjoint from J. We can construct R so that R and $-R$ intersect only at 0. Fix such an R. Then $\mathbb{C} \setminus (R \cup -R)$ has two components S_0' and S_1', where we may assume that $f(0) \in S_0'$. If we let $S_* = \{0\}$ and $S_i = S_i' \cap J$, then $\{S_*, S_0, S_1\}$ is a partition of J from which we may define itineraries.

Let τ be the itinerary of $f(0)$ with respect to this partition, called the *kneading sequence* of J. Let T be the smallest subcontinuum of J containing the orbit of 0. If the orbit of T is finite, then T will be a tree, called the *Hubbard Tree* of J. (See [BS] for more on Hubbard Tree maps.) Let f, c, J, $\{S_*, S_0, S_1\}$, T be fixed for the remainder of this section. Note that in the case we are covering 0 is not periodic because it would then be attracting, and would not be in the boundary of J.

Theorem 2.1. *The map $f|J$ has the unique itinerary property on J.*

Proof. In [Bal2], Theorem 2.5, it was shown that $f|J$ was conjugate to a function which is known to have the unique itinerary property. □

Theorem 2.2. *Let $\hat{J}$ and $\hat{T}$ be the inverse limits of J and T respectively (with respect to the obvious restrictions of f). Then every indecomposable subcontinuum of $\hat{J}$ is also a subcontinuum of $\hat{T}$. In particular, if $J \neq T$, then $\hat{J}$ is decomposable. Furthermore, $\hat{T}$ is the smallest subcontinuum of $\hat{J}$ which contains all indecomposable subcontinua of $\hat{J}$.*

Proof. Let $\hat{C}$ be a subcontinuum of $\hat{J}$ which is not a subcontinuum of $\hat{T}$. We need to show that $\hat{C}$ is decomposable. Let $C_n = \pi_n(\hat{C})$. Since $\hat{C}$ contains a point not in $\hat{T}$, there is an integer N such that $C_n \setminus T$ is nonempty for all $n \leq N$.

Case 1: For all $n \leq N$, $C_n \setminus T$ has exactly one component. Let U be the unique component of $C_N \setminus T$. Then for all $n < N$, $f^{n-N}(U) \cap C_n$ has only one component, for otherwise $\overline{C_n \setminus T}$ would have more than one component. Thus, if we let $A_n = \overline{f^{n-N}(U) \cap C_n}$ for $n \leq N$ and $A_n = f^{n-N}(A_N)$ for $n > N$, we have that for $n < N$, $f : A_n \to A_{n+1}$ is a homeomorphism, and therefore the inverse limit $\hat{A}$ of the A_n's is a dendrite. If $\hat{A}$ is all of $\hat{C}$, then it is decomposable and we are done. Otherwise, $\hat{A}$ has exactly one point which is a limit point of $\hat{C} \setminus \hat{A}$, so $\hat{A}$ and $\overline{\hat{C} \setminus \hat{A}}$ are two nondegenerate proper subcontinua of $\hat{C}$ whose union is $\hat{C}$.

Case 2: For some $M < N$, $C_M \setminus T$ has at least two components. Fix such an M, and let U and V be two of the components of $C_M \setminus T$. For $n \leq M$, let $A_n = C_n \setminus f^{n-M}(U)$, and let $A_n = f^{n-M}(A_M)$ for $n > M$. Note that A_M is connected, being the result of "pruning" the one piece U from C_M. Similarly, for $n < M$, A_n is the result of "pruning" no more than 2^{n-M} pieces from C_n, and is also connected. Thus, if $\hat{A}$ is the inverse limit of the A_n's with respect to f, then $\hat{A}$ is a nondegenerate proper subcontinuum of $\hat{C}$. Similarly, if we let $B_n = C_n \setminus f^{n-M}(V)$ for $n \leq M$, and let $B_n = f^{n-M}(B_M)$ for $n > M$, then the inverse limit $\hat{B}$ of the B_n's is a nondegenerate proper subcontinuum of $\hat{C}$ such that $\hat{A} \cup \hat{B} = \hat{C}$.

Thus, $\hat{C}$ is decomposable. The rest of the theorem is an immediate consequence of the previous results. □

The following easy result shows that the points of $\hat{J} \setminus \hat{T}$ are connected to $\hat{T}$ in a relatively simple way.

Proposition 2.3. *If $\hat{a} \in \hat{J} \setminus \hat{T}$, there is a smallest subcontinuum $\hat{A}$ containing $\hat{T} \cup \{\hat{a}\}$ such that $\hat{A} \setminus \hat{T}$ is a ray with initial point $\hat{a}$.*

Proof. Let A_n be the smallest tree containing $T \cup \{a_n\}$. Then $f(A_n) = A_{n+1}$, and it is routine to check that if $\hat{A}$ is the inverse limit of the A_n's, then $\hat{A}$ is as desired. □

The "open" end of the ray $R = \hat{A} \setminus \hat{T}$ can limit on either a point or a nondegenerate continuum. It will be a point if and only if there is an integer N such that for all $n \leq N$, $f(A_n \setminus T) = A_{n+1} \setminus T$.

One well known feature of inverse limits of Markov tree maps is the common presence of two-sided rays which limit on a nondegenerate continuum at both ends, i.e., continuous images of $\mathbb{R}$ which cannot be continuously extended either at $+\infty$ or $-\infty$. In fact, all but finitely many of the uncountably many arc-components of $\hat{T}$ are two-sided rays, as is more precisely indicated by the following theorem.

Theorem 2.4. *Let P be the Markov set of T, and let $\hat{P}$ be the inverse limit of P with respect to f. Then $\hat{P}$ consists of finitely many periodic points of $\hat{f}$, and every arc-component of $\hat{T}$ which misses $\hat{P}$ is a two-sided ray.*

Proof. It is easy to see that if $\hat{p} \in \hat{P}$, then p_n cannot be a strictly preperiodic element of P, so $\hat{P}$ is the same as $\hat{P}'$, where P' is the set of periodic points of P (on which f is one-to-one). Let $\hat{A} \subseteq \hat{T}$ be a nondegenerate subcontinuum of $\hat{T}$ which contains no element of $\hat{P}$. Let $A_n = \pi_n(\hat{A})$. Then since no element of $\hat{P}$ is in $\hat{A}$, there is an integer N such that $A_n \cap P = \emptyset$ for all $n \leq N$. Thus each A_n for $n \leq N$ is a proper subset (at both ends) of a P-basic interval I_n such that $I_{n+1} \subseteq f(I_n)$ and $f|A_n$ is one-to-one for such n, and therefore A is an arc. Thus, if we let $B_N = I_N$, and define B_n for $n < N$ such that $A_n \subseteq B_n \subseteq I_n$ and $f(B_n) = B_{n+1}$, with $B_n = f^{n-N}(B_N)$ for $n > N$, then the inverse limit of the B_n's is a subcontinuum $\hat{B}$ which is an arc extending $\hat{A}$ at both ends. Thus no arc-component missing $\hat{P}$ can have either an endpoint or a subcontinuum which is not an arc, and therefore every arc-component missing $\hat{P}$ must be a two-sided ray. □

Definition 2.5. Let a two-sided ray in a space X be called *maximal* if it cannot be extended to a larger ray (either one-sided or two-sided), i.e., it has no single point limit in X at either the $+\infty$ or $-\infty$ end.

Proposition 2.6. *Let $\hat{A}$ be an arc in $\hat{J}$ which has endpoints in $\hat{T}$. Then $\hat{A} \subseteq \hat{T}$.*

Proof. If $\hat{a}$ and $\hat{b}$ are the endpoints of $\hat{A}$, then there is an integer N so that $A_n = [a_n, b_n] \subseteq T$ for $n \leq N$. □

Theorem 2.7. *Let D be a dendrite, let $h : D \to D$ be continuous, and suppose that D' is a subdendrite of D such that $h(D') = D'$ and that h is locally one-to-one on $D \setminus D'$. Let $\hat{D} = (D, h)$, $\hat{D}' = (D', h|D')$. Then every maximal two-sided ray in $\hat{D}$ is a subset of $\hat{D}'$.*

Proof. Let $\langle \hat{A}^{(k)} = [\hat{a}^{(k)}, \hat{c}] : k \in \omega \rangle$ and $\langle \hat{B}^{(k)} = [\hat{b}^{(k)}, \hat{c}] : k \in \omega \rangle$ be two strictly increasing nested sequence of arcs with $\hat{A}^{(k)} \cap \hat{B}^{(k)} = \{\hat{c}\} \in \hat{D} \setminus \hat{D}'$ for all k, so that $R_A = \bigcup_{k\in\omega} \hat{A}^{(k)}$ and $R_B = \bigcup_{k\in\omega} \hat{B}^{(k)}$ are two rays with initial point $\hat{c}$ such that $R_A \cap R_B = \{\hat{c}\}$. Let $A_n^{(k)} = \pi_n(\hat{A}^{(k)})$, $B_n^{(k)} = \pi_n(\hat{B}^{(k)})$, $a_n^{(k)} = \pi_n(\hat{a}^{(k)})$, $b_n^{(k)} = \pi_n(\hat{b}^{(k)})$. Since $\hat{c} \notin \hat{D}'$, there is an integer N such that $c_n \in D \setminus D'$ for all $n \leq N$. Thus, since h is locally one-to-one on $D \setminus D'$, we must have that for all k and all $n \leq N$, at least one of $A_n^{(k)}$ and $B_n^{(k)}$ is disjoint from D'. Without loss of generality, $A_n^{(k)}$ is disjoint from D' for all $n \leq N$. But then $h : A_n^{(k)} \to A_{n+1}^{(k)}$ is a homeomorphism for all k and all $n \leq N$, so for each $n \leq N$, the $A_n^{(k)}$'s ($k \in \omega$) form a strictly increasing nested sequence of arcs in $D \setminus D'$. Thus, if we let $a_n = \lim\{a_n^{(k)}\}$, then $\hat{a}$ is the unique limit of the open end of the ray R_A, and therefore $R_A \cup R_B$ is not a maximal two-sided ray. □

3. The case of "circular" Julia Sets

If c is in the interior, or on the cusp, of one of the infinitely many "cardioids" of the Mandelbrot Set, then the Julia Set J of $f(z) = z^2 + c$ contains infinitely many circles (or, in the case of the largest cardiod, consists of a single circle). Nevertheless, there is a dendrite-like structure to J of which we can take advantage, for shrinking each of the circles of J to points gives a quotient space D which is a dendrite, and induces a map $g : D \to D$ which is closely related to f, so that the inverse limit structure of $\hat{J}$ is closely related to the structure of $\hat{D}$, which can be exploited to get results similar to the previous section. For the remainder of the section, fix a function $f(z) = z^2 + c$, where c is in the interior or at the cusp of one of the smaller "cardioids" of the Mandelbrot Set (so that the Julia Set is more than just a single circle), and let J be the Julia Set of f. The main connection between J and D are known, and are outlined in the following theorem.

Theorem 3.1. *Let f be the complex map $f(z) = z^2 + c$, and assume that c is in the interior or at the cusp of one of the small cardioids of the Mandelbrot Set (equivalently, has an attracting or indifferent periodic point of some period $p > 1$ such that the corresponding kneading sequence is not an n-tupling). Let J be the Julia Set of f. Then J contains circles, any two of which are disjoint. Let D be the quotient space obtained by shrinking all circles of J to points, with quotient map $q : J \to D$. Then D is a dendrite, and if $q(x) = q(y)$, then $q(f(x)) = q(f(y))$, so that there is a well defined continuous function $g : D \to D$ such that $q \circ f = g \circ q$, i.e., q is a semiconjugacy between f and g. Let $Z \subseteq D$ be the set of all x such that $q^{-1}(x)$ is a circle, and let $C_x = q^{-1}(x)$. Then*

1. *$g(Z) = Z$, $g(D \setminus Z) = D \setminus Z$, and Z is countable.*
2. *The function g has a unique turning point $t \in Z$, and $D \setminus \{t\}$ can be partitioned into two sets S_0 and S_1 such that g has the unique itinerary property with respect to $\{\{t\}, S_0, S_1\}$.*
3. *t is periodic with respect to g with period p.*
4. *If $x \in Z \setminus \{t\}$, then $f|C_x$ is one-to-one.*
5. *$f|C_t$ is a covering map of degree 2.*
6. *$f^p(C_t) = C_t$, and $f^p|C_t$ is conjugate to the angle doubling map on the circle.*
7. *For each $x \in Z$, there is an $n \in \omega$ such that $f^n|C_x$ is a homeomorphism from C_x onto C_t.*

Proof. See [Ba2], especially Theorem 3.24 and Theorem 4.11. □

Let T be the smallest subtree of D containing the orbit of t, and let $K = q^{-1}(T)$. J, f, D, g, t, K, and T are now fixed for the remainder of this section. The function $g : T \to T$ is the *Hubbard Tree* map for the Julia Set J, and the itinerary of $g(t)$ with respect to the function g and the partition $\{\{t\}, S_0, S_1\}$ is the *kneading sequence* of J. The following is easy to prove.

Proposition 3.2. *K is connected.*

Let $\hat{J}$ and $\hat{K}$ be the inverse limits of J and K respectively with respect to the relevant restrictions of f, with corresponding shift map $\hat{f} : \hat{J} \to \hat{J}$. Let $\hat{D}$ and $\hat{T}$ be the inverse limits of D and T respectively with respect to the relevant restrictions of g, with corresponding shift map $\hat{g} : \hat{D} \to \hat{D}$. Then clearly $\hat{K}$ is a subspace of $\hat{J}$ and $\hat{T}$ is a subspace of $\hat{D}$. The relation between the former two and the latter two can be given by a quotient map induced from q.

Proposition 3.3. *The map $\hat{q} : \hat{J} \to \hat{D}$ defined by $\hat{q}(\langle x_n \rangle) = \langle q(x_n) \rangle$ is a well defined quotient map such that $\hat{q} \circ \hat{f} = \hat{g} \circ \hat{q}$.*

D and $\hat{D}$ are very similar to the Julia Sets which were dendrites discussed in the previous section, so that the following corresponding result is true, with a nearly identical proof to the proofs given in section 2.

Theorem 3.4. *$\hat{T}$ contains an indecomposable continuum (in some, but not all, cases is itself an indecomposable continuum), and it is the smallest subcontinuum of $\hat{D}$ which contains all indecomposable subcontinua of $\hat{D}$. In particular $\hat{D}$ is decomposable. Also, if $R \subseteq \hat{D}$ is a continuous one-to-one image of $\mathbb{R}$ having no endpoints, then $R \subseteq \hat{T}$.*

On the other hand, it is clear that $\hat{J}$ is quite different from $\hat{D}$. In particular, $\hat{J}$ contains circles and solenoids.

Definition 3.5. *A* standard solenoid *is any space homeomorphic to the inverse limit of the angle doubling map on the circle.*

It is well known that the standard solenoid is an indecomposable continuum, all of whose proper subcontinua are arcs.

Theorem 3.6. (1) *$\hat{J}$ contains exactly p standard solenoids.*

(2) *$\hat{J}$ contains uncountably many circles. In fact, if $\hat{E}$ is a circle in $\hat{J}$, and $\hat{U} \subseteq \hat{J}$ is an open set containing $\hat{E}$, then $\hat{U}$ contains a subset homeomorphic to the circle cross a Cantor Set.*

Proof. Let $\hat{E}$ be a subcontinuum such that for every n, $E_n = \pi_n(E)$ is a circle. Then for each n there is an $x_n \in Z$ such that $E_n = C_{x_n}$. Then there are two cases.

Case 1: There are infinitely many negative n such that $x_n = t$. Then for some i, $0 \leq i \leq p-1$, $x_n = t$ iff $n - i$ is divisible by p. Since $C_{x_{n+1}} = f(C_{x_n})$, $x_{n+1} = g(x_n)$, and the rest of the x_n's are determined by i. Thus, there are exactly p possibilities here. Since f is a covering map of degree 2 for infinitely many negative n and a homeomorphism for all other negative n, $\hat{E}$, the inverse limit of the E_n's, is a standard solenoid.

Case 2: There is an integer N such that $x_n \neq t$ for all $n \leq N$. Then $f|E_n$ is a circle homeomorphism for all $n \leq N$, so $\hat{E}$ is a circle. Suppose that $\hat{U}$ is an open set containing $\hat{E}$. Then there is an $n < N$ such that $\pi_n^{-1}(E_n) \subseteq \hat{U}$. Then for each $k < n$, $f^{-k}(E_n)$ consists of 2^k disjoint circles, with f mapping $f^{-k}(E_n)$ two-to-one onto $f^{-k+1}(E_n)$, so $\pi_n^{-1}(E_n)$ is homeomorphic to a circle cross a Cantor Set.

It is easy to see that no subset of $\hat{J}$ other than the ones described above can be homeomorphic to either a circle or a solenoid. □

Lemma 3.7. *If X is a subcontinuum of J, and Y is a subcontinuum of $q(X)$, then $q^{-1}(Y) \cap X$ is connected.*

Proof. Let $x, y \in q^{-1}(Y) \cap X$. Since X is also locally connected, there is a one-to-one continuous function $h : [0,1] \to X$ such that $h(0) = x$ and $h(1) = y$. Then since $q : J \to D$ is monotone, $q \circ h$ is also monotone, so $q(h([0,1])) = [q(x), q(y)] \subseteq Y$. Thus $[x, y] \subseteq q^{-1}(Y)$, and $q^{-1}(Y) \cap X$ is path connected and therefore connected. □

Lemma 3.8. *If $\hat{X}$ is a subcontinuum of $\hat{J}$, and $\hat{Y}$ is a subcontinuum of $\hat{q}(\hat{X})$, then $\hat{q}^{-1}(\hat{Y}) \cap \hat{X}$ is connected.*

Proof. Let $X_n = \pi_n(\hat{X})$, and $Y_n = \pi'_n(\hat{Y})$. Then for each n, Y_n is a subcontinuum of $q(X_n)$, and therefore $W_n = q^{-1}(Y_n) \cap X_n$ is connected. Since $f(W_n) = W_{n+1}$, the inverse limit of the W_n's with respect to f is a well defined subcontinuum $\hat{W}$ of $\hat{X}$. We claim that $\hat{W} = \hat{q}^{-1}(\hat{Y}) \cap \hat{X}$.

Thus, suppose that $\hat{w} \in \hat{W}$. Then clearly $\hat{w} \in \hat{X}$. Since $w_n \in q^{-1}(Y_n)$ for all n, $q(w_n) \in Y_n$ for all n. Therefore $\hat{q}(\hat{w}) = \langle q(w_n) \rangle \in \hat{Y}$ and therefore $\hat{w} \in \hat{q}^{-1}(\hat{Y})$.

In the other direction, if $\hat{w} \in \hat{q}^{-1}(\hat{Y}) \cap \hat{X}$, then $\hat{q}(\hat{w}) \in \hat{Y}$, so $q(w_n) \in Y_n$ for all n. Thus, $w_n \in q^{-1}(Y_n)$. Since $\hat{w} \in \hat{X}$ implies that $w_n \in X_n$, $w_n \in W_n$ for all n, i.e., $\hat{w} \in \hat{W}$. □

Corollary 3.9.
(1) *If $\hat{X}$ is an indecomposable subcontinuum of $\hat{J}$ and $\hat{q}(\hat{X})$ contains more than one point, then $\hat{q}(\hat{X})$ is also indecomposable.*
(2) *If $\hat{X}$ is a subcontinuum of $\hat{J}$, and $\hat{q}(\hat{X})$ is a decomposable subcontinuum of $\hat{D}$, then $\hat{X}$ is decomposable.*

Proof. To prove (2), suppose that $\hat{q}(\hat{X})$ is decomposable, say $\hat{q}(\hat{X}) = \hat{A} \cup \hat{B}$, where $\hat{A}$ and $\hat{B}$ are proper subcontinua of $\hat{q}(\hat{X})$. Then by Lemma 3.8, $\hat{q}^{-1}(\hat{A}) \cap \hat{X}$ and $\hat{q}^{-1}(\hat{B}) \cap \hat{X}$ are proper subcontinua of $\hat{X}$ whose union is $\hat{X}$. (1) is clearly equivalent to (2). □

Theorem 3.10. *Every indecomposable subcontinuum of $\hat{J}$ is a subset of $\hat{K}$.*

Proof. Let $\hat{X}$ be an indecomposable subcontinuum of $\hat{J}$.

Case 1: $\hat{q}(\hat{X})$ is a singleton $\{\hat{z}\}$. Either $z_n \in Z$ for all n or $z_n \notin Z$ for all n, but the latter is impossible, because then $\hat{q}^{-1}(\hat{z})$ would be a singleton, contradicting indecomposability of $\hat{X}$. Thus $z_n \in Z$ for all n and therefore $X_n = \pi_n(\hat{X}) \subseteq C_{z_n}$ for all n. The only way this could give an indecomposable continuum is if $\hat{X}$ is a solenoid, but the p solenoids contained in $\hat{J}$ are all subsets of $\hat{K}$.

Case 2: $\hat{q}(\hat{X})$ contains more than one point. Then by Corollary 3.9, $\hat{q}(\hat{X})$ is indecomposable, and therefore by Theorem 3.4, $\hat{q}(\hat{X}) \subseteq \hat{T}$. Thus $\hat{X} \subseteq \hat{q}^{-1}(\hat{T}) = \hat{K}$. □

In analogy with Theorem 2.2, an obvious question would be whether or not $\hat{K}$ is the smallest subcontinuum of $\hat{J}$ containing all indecomposable subcontinua of $\hat{J}$. The answer is that it depends on which Julia Set J we are considering. The following example shows that there might be a proper subcontinuum of $\hat{K}$ containing all of the indecomposable continua.

Example 3.11. Let c be a real number such that $f(z) = z^2 + c$ has an attracting period six orbit that is Sharkovsky minimal, i.e., $f|\mathbb{R}$ has no points of odd period other than a fixed point. It is well known that such c exist. Then T will be an interval $[g(t), g^2(t)]$ with turning point t, with the orbit of $g^6(t) = t$ and the fixed point x appearing on T in the order $g(t) < g^5(t) < g^3(t) < x < g^4(t) < t < g^2(t)$ (so that g maps $[g(t), x]$ one-to-one onto $[x, g^2(t)]$, and $[x, g^2(t)]$ onto $[g(t), x]$). Then $g^2|[g(t), g^3(t)]$ and $g^2|[g^4(t), g^2(t)]$ are both l.e.o., and $\hat{T}$ has two indecomposable subcontinua $\hat{A}$ and $\hat{B}$, with an arc winding between them. The indecomposable subcontinua of $\hat{J}$ will be the six solenoids implied by Theorem 3.6 plus all indecomposable continua (uncountably many of them) contained in either $\hat{q}^{-1}(\hat{A})$ or $\hat{q}^{-1}(\hat{B})$. Starting with the continuum $\hat{K}$, pick a random $y \in Z \cap [g^3(t), g^4(t)]$ and remove a part U of the circle C_y from K, being careful that $K \setminus U$ is still a continuum. Then $L = K \setminus \bigcup_{n\in\omega} f^{-n}(U)$ will still be a continuum, invariant under f, and all of the pieces removed will be from $q^{-1}[g^3(t), g^4(t)]$, so that $\hat{L}$ will still contain all indecomposable subcontinua of $\hat{J}$. Removing a different piece U' could give a different continuum $L' = K \setminus \bigcup_{n\in\omega} f^{-n}(U')$, so that $\hat{L}'$ also contains all indecomposable subcontinua, but $\hat{L} \cap \hat{L}'$ is not connected. Thus, there is no *smallest* subcontinuum containing all of the indecomposable subcontinua (although there will be *minimal* such continua).

For the remainder of this section, we add the additional hypothesis that the parameter value c is such that $f(z) = x^2 + c$ has an attracting period orbit of period p and the corresponding tree map $g : T \to T$ is locally eventually onto. There are many parameter values for which this occurs (see [Bal1], Theorem 4.13, for a combinatorial characterization of the kneading sequences associated with such c). In that case, the continuum $\hat{K}$ will be the smallest continuum containing all of the indecomposable subcontinua of $\hat{J}$. To start, it helps to see why $\hat{K}$ itself is not indecomposable.

Theorem 3.12. *$\hat{K}$ is decomposable.*

Proof. Pick $x \in Z$ which is not periodic with respect to g (i.e., not in the orbit of t), noting that x is not an endpoint of T. The circle $C_x \subseteq K$ has a disconnected interior having two components. Let U and V be the two components of the interior of K, and let $A_0 = K \setminus U$, $B_0 = K \setminus V$. Let $A_n = f^n(A_0)$, $B_n = f^n(B_0)$. Then A_n and B_n are connected subsets of K, and if $\hat{A}$ and $\hat{B}$ are the inverse limits of the A_n's and B_n's respectively, then $\hat{A}$ and $\hat{B}$ are two proper subcontinua of $\hat{K}$ whose union is $\hat{K}$. □

To construct an indecomposable subcontinuum of $\hat{K}$ other than the p solenoids guaranteed by Theorem 3.6, let $C_i = C_{g^i(t)}$, $0 \leq i \leq p-1$, and let $\mathcal{M}$ be the set of all minimal subcontinua of K which contain each C_i, $0 \leq i \leq p-1$. Note that if $A \in \mathcal{M}$, then $(f|K)^{-1}(A)$ is a continuum. $(f|K)^{-1}(A)$ will not be an element of $\mathcal{M}$, because if $x \in Z$ such that x is not in the orbit of t but $f(x)$ is in the orbit of t (and there will always be at least one such x), then $(f|K)^{-1}(A)$ contains the entire circle C_x and therefore fails the minimality property. However, removing half of each such C_x from $(f|K)^{-1}(A)$ will give a $B \in \mathcal{M}$ such that $f(B) = A$.

Lemma 3.13. *Suppose that $A_n \in \mathcal{M}$ for all $n \leq N$ and $A_{n+1} = f(A_n)$ for all n. Let $\hat{A}$ be the inverse limit of the A_n's. Then $\hat{A}$ is indecomposable.*

Proof. Let $\hat{X}$ and $\hat{Y}$ be two continua whose union is $\hat{A}$. Since $g : T \to T$ is l.e.o., $\hat{T}$ is indecomposable. Thus, since $\hat{q}(\hat{X}) \cup \hat{q}(\hat{Y}) = \hat{T}$, one of $\hat{q}(\hat{X})$ and $\hat{q}(\hat{Y})$ must be all of $\hat{T}$. Without loss of generality, $\hat{q}(\hat{X}) = \hat{T}$. Let $X_n = \pi_n(\hat{X})$ for each n. Then $q(X_n) = T$ for all n. The turning point t of g is a cutpoint of T, so $C_0 = C_t$ can be divided into two closed sets V and W such that $f(V) = f(W) = C_1$ and every continuum $B \subseteq K$ such that $q(B) = T$ contains either V or W. If n is an integer and i is an integer such that $1 \leq i \leq p$, then X_{n-i} must contain either V or W, and therefore X_n must contain $f^i(V) = f^i(W) = C_i$ (letting $C_p = C_0 = C_t$). Thus, $X_n \subseteq A_n$ is a continuum containing all C_i's for $0 \leq i \leq p-1$, and the minimality of $A_n \in \mathcal{W}$ for $n \leq N$ implies that $X_n = A_n$ for all $n \leq N$ (and therefore for all n). Thus $\hat{X} = \hat{A}$. □

Corollary 3.14. *If $g : T \to T$ is l.e.o., then $\hat{K}$ is the smallest continuum containing all of the indecomposable subcontinua of $\hat{J}$.*

Proof. If $\hat{a} \in \hat{K}$, then $a_0 \in K$, and there is an $A_0 \in \mathcal{M}$ such that $a_0 \in A_0$. For $n \in \omega$, let $A_{n+1} = f(A_n)$. For negative n define A_n by backwards induction. If $A_n \in \mathcal{M}$ has been defined for some $n \leq 0$ such that $a_n \in A_n$, then $a_{n-1} \in f^{-1}(A_n)$, and it is easy to see that $f^{-1}(A_n)$ contains an $A_{n-1} \in \mathcal{M}$ such that $a_{n-1} \in A_{n-1}$. Then $\hat{A}$, the inverse limit of the A_n's, is indecomposable and $\hat{a} \in \hat{A}$. □

4. Distinguishing inverse limits of Julia Sets

One obvious question would be whether or not the inverse limits of different Julia Sets are homeomorphic. In the cases discussed here, Julia Sets having the same kneading sequence for their corresponding Hubbard Trees have conjugate maps $f|J : J \to J$ and therefore homeomorphic inverse limits. Thus, it is natural to ask the following question.

Question 4.1. *If $f_c(z) = z^2 + c$ and $f_d(z) = z^2 + d$ are two polynomial maps with connected Julia Sets J_c and J_d, and different kneading sequences for their corresponding Hubbard Trees, must $\hat{J}_c$ and $\hat{J}_d$ necessarily be nonhomeomorphic?*

Of course, the inverse limit of a "dendritic" Julia Set and the inverse limit of a "circular" Julia Set are obviously very different, but the case where they are both dendrites or both circular is less obvious. The following result shows that this question is related to the same question about the inverse limits of the Hubbard Trees.

Theorem 4.2. *If $f_c(z) = z^2 + c$ and $f_d(z) = z^2 + d$ are two polynomial maps with Julia Sets J_c and J_d which are both dendrites, and the corresponding inverse limits $\hat{J}_c$ and $\hat{J}_d$ are homeomorphic, then $\hat{T}_c$ and $\hat{T}_d$, the inverse limits of the corresponding Hubbard Trees T_c and T_d, are also homeomorphic.*

Proof. As proven above in Theorem 2.2, $\hat{T}_c$ is the smallest subspace of $\hat{J}_c$ containing all indecomposable subcontinua of $\hat{J}_c$, and $\hat{T}_d$ is the smallest subspace of $\hat{J}_d$ containing all indecomposable subcontinua of $\hat{J}_d$. Since the description "the smallest subspace containing all indecomposable subcontinua" is a topological property, and $\hat{J}_c$ is homeomorphic to $\hat{J}_d$, $\hat{T}_c$ is homeomorphic to $\hat{T}_d$. □

Theorem 4.3. *If $f_c(z) = z^2 + c$ and $f_d(z) = z^2 + d$ are two polynomial maps having attracting periodic orbits whose kneading sequences are not n-tuplings, the corresponding inverse limits $\hat{J}_c$ and $\hat{J}_d$ are homeomorphic, and $g_c : T_c \to T_c$ and $g_d : T_d \to T_d$ are the corresponding Hubbard Tree maps, then the inverse limits $\hat{T}_c$ and $\hat{T}_d$, are also homeomorphic.*

Proof. Since $q_c : J_c \to D_c$ and $q_d : J_d \to D_d$ both shrink all circles of J_c and J_d to points, it is easy to see that $\hat{q}_c : \hat{J}_c \to \hat{D}_c$ and $\hat{q}_d : \hat{J}_d \to \hat{D}_d$ are the results of shrinking all circles and solenoids of $\hat{J}_c$ and $\hat{J}_d$ to points. Therefore, $\hat{D}_c$ and $\hat{D}_d$ are homeomorphic. Since $\hat{T}_c$ is the smallest subspace of $\hat{D}_c$ containing all indecomposable subcontinua of $\hat{D}_c$, and $\hat{D}_d$ is the smallest subspace of $\hat{D}_d$ containing all indecomposable subcontinua of $\hat{J}_d$, it follows as in Theorem 4.2 that $\hat{T}_c$ is homeomorphic to $\hat{T}_d$. □

In the case where T is an arc, inverse limits of maps with one turning point have been studied extensively (e.g., [BBS], [BaD], [BM], [BJKK], [BKRS], [Br], [I], [K1], [K2], [RS], [S1], [S2]). The *slope λ tent map* $f_\lambda : [0,1] \to [0,1]$ is defined by

$$f_\lambda(x) = \begin{cases} \lambda x, & 0 \le x \le \frac{1}{2} \\ \lambda(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

Ingram's Conjecture, recently proven by Barge, Bruin, and Štimac [BBS], states that the inverse limits of $[0,1]$ with respect to the tent maps are nonhomeomorphic for distinct $\lambda \in [1,2]$. For $\sqrt{2} < \lambda \le 2$, the *slope λ core tent map* is the map $f_\lambda|[f_\lambda^2(\frac{1}{2}), f_\lambda(\frac{1}{2})]$. The core tent maps are l.e.o., and are conjugate to Hubbard Trees for quadratic Julia Sets. The "core" version of Ingram's Conjecture is apparently still unsolved, but it is known to be true in the case where the turning point is periodic or preperiodic.

Theorem 4.4. *Let f_λ and f_ν be two tent maps, with $\lambda \neq \nu$ both in $[\sqrt{2}, 2]$, and let $f_\lambda|I_\lambda$ and $f_\nu|I_\nu$ be their core restrictions.*

(1) *Suppose that the turning point $\frac{1}{2}$ is periodic in both cases. Then the inverse limits of $f_\lambda|I_\lambda$ and $f_\nu|I_\nu$ are nonhomeomorphic.* (Kailhofer [K2])
(2) *Suppose that the turning point $\frac{1}{2}$ is preperiodic in both cases. Then the inverse limits of $f_\lambda|I_\lambda$ and $f_\nu|I_\nu$ are nonhomeomorphic.* (Štimac [S2])

These results have recently been extended to simple trees:

Theorem 4.5. *If $f_0 : T_0 \to T_0$ and $f_1 : T_1 \to T_1$ are l.e.o. tree maps with only one turning point which is periodic or preperiodic, on trees T_0 and T_1 having only one branch point, then they have homeomorphic inverse limits if and only if they have the same kneading sequence.*

Proof. The periodic case was the main result of [Bal3], and the preperiodic case will be covered in [Bal4]. □

From these results and the other results in this paper, it follows that the inverse limits of Julia Sets can be distinguished in many cases where the Hubbard Tree is an arc.

Theorem 4.6. *Let $f_c(z) = z^2 + c$, $f_d(z) = z^2 + d$, where f_c and f_d either have a finite critical orbit or an attracting periodic point, and suppose that the Hubbard Tree maps associated with the Julia Sets J_c and J_d are locally eventually onto maps of trees with no more than one branch point. Then the inverse limits $\hat{J}_c$ and $\hat{J}_d$ are homeomorphic if and only if the Hubbard Trees have the same kneading sequence.*

Proof. Hubbard Trees with the same kneading sequence give conjugate Julia Set maps, which in turn give homeomorphic inverse limits. In the other direction, suppose that $\hat{J}_c$ and $\hat{J}_d$ are homeomorphic. The l.e.o. property guarantees that we are not dealing with period n-tuplings (period doublings in the case of arcs). A Hubbard Tree map which is l.e.o. is also transitive, and therefore, by a result of Parry [P], conjugate to a constant-slope map of a tree. If the tree is an interval, this map can in turn, with appropriate scaling, be regarded as a restriction of a tent map to its core, with expansion factor in $[\sqrt{2}, 2]$. By Theorems 4.2 and 4.3, $\hat{T}_c$ and $\hat{T}_d$ are homeomorphic, and therefore by Theorem 4.4, the tent maps conjugate to $\hat{T}_c$ and $\hat{T}_d$ have the same expansion factor λ, and therefore the same kneading sequence. If the tree has a single branch point, use the same argument with Theorem 4.5 □

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