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SOME FURTHER INVESTIGATIONS OF OPEN COVERS AND SELECTION PRINCIPLES USING IDEALS

DEBRAJ CHANDRA AND PRATULANANDA DAS

Abstract. Following the methods of a very recent work of the second author [5] we make a new and more general approach to certain results on open covers and selection principles in hyperspaces and function spaces by using the notion of ideals and ideal convergence, and investigate some of its consequences which automatically extends similar classical results (where finite sets are used) as well as recent statistical variants studied by Di Maio and Kočinac [6].

1. Introduction

We start by recalling the definition of natural or asymptotic density as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(n)$ denotes the set $\{k \in K : k \leq n\}$ and $|K(n)|$ stands for the cardinality of the set $K(n)$. The natural or asymptotic density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$$

provided the limit exists.

It can be mentioned in this context that using the idea of asymptotic density, the idea of convergence of a real sequence had been extended to statistical convergence by Fast [11] (see also Schoenberg [27]) as follows: A sequence $\{x_n\}_{n\in N}$ of points in a metric space (X, ρ) is said to be statistically convergent to ℓ if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in N : d(x_k, \ell) \geq \epsilon\}$ has natural density zero. A lot of investigations have been done on this convergence and its topological consequences after

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the initial works by Fridy [12] and Šalat [24] (for more references see [2], [20], [21]).

In particular, very recently Di Maio and Kočinac [6] introduced the concept of statistical convergence in topological spaces as well as uniform spaces and established the topological nature of this convergence. Most importantly they also offered very interesting applications of the notion of asymptotic density to open covers of topological spaces and the selection principles related to the covers. In the process they obtained statistical analogues of some kinds of open covers and related selection principles. Study of open covers and selection principles and their interrelationship has a long illustrious history and readers interested in selection principles and its recent developments can consult the papers [1], [6 - 9], [13 - 19], [25], [26], [28], [29] where many more references can be found.

On the other hand, in [20] an interesting generalization of the notion of statistical convergence was proposed. Namely it is easy to check that the family $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible ideal of $\mathbb{N}$ (recall [20], [21] that if X is a nonempty set, then a family I of subsets of X is said to be an *ideal* in X if (i) $A, B \in I$ implies $A \cup B \in I$ and (ii) $A \in I$, $B \subset A$ implies $B \in I$. A nonempty family F of subsets of X is called a *filter* on X if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$, (iii) $A \in F$ and $A \subset B$ implies $B \in F$. I is called *non-trivial* if $I \neq \{\phi\}$ and $X \notin I$. If I is a proper nontrivial ideal then the family of sets $F(I) = \{M \subset X : \text{there exists } A \in I : M = X \setminus A\}$ is a filter in X. It is called the *dual filter* of the ideal I. A proper ideal I is called *admissible* if $\{x\} \in I$ for each $x \in X$. (Such ideals are also called *free* ideals.) Thus one may consider an arbitrary ideal I of $\mathbb{N}$ and define I-convergence of a sequence by replacing the sets of density zero by the members of the ideal. Following the general line of [20], ideals were used to study sequences in topological spaces [21], to study nets in topological and uniform spaces ([22], [3], [4]). In particular some results of [6] concerning statistical convergence were proved in more general form in [21] and further in [22] (one can also see [3] where some possible solutions of an open problem posed in [6] have been discussed).

In [5] the notion of ideals were applied to certain types of open covers in topological and uniform spaces and selection principles related to those covers and ideal analogues of some kinds of open covers and related selection principles were presented which automatically extended the results of [6] and presented those results in much more general form. In this context it should be noted that a similar approach was made using filters and in particular semi-filters. In [23] the dual approach of looking at filters instead of ideals is explored (see [5] for more references for this type of work).

In this short paper, we continue the investigation in line of [5] and apply the notion of ideals to certain types of open covers in topological spaces and selection principles related to those covers primarily in function spaces and hyperspaces which automatically extend the results of [6] and present those results in much more general form.

2. Main Results

2.1. **Applications of ideals to open covers.** We reproduce some fundamental definitions and properties from [5] for easy reference which will be needed throughout the paper. Throughout the paper (X, τ) stands for a Hausdorff topological space unless otherwise mentioned. $\mathbb{F}(X)$, $\mathbb{CL}(X)$, $\mathbb{K}(X)$ stand for the classes of nonempty finite sets, closed sets and compact sets of X respectively.

For two nonempty classes of sets $\mathbf{A}$ and $\mathbf{B}$ of an infinite set X, following [25] we define (see [5], [6]) :

$S_1(\mathbf{A}, \mathbf{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathbf{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each n and $\{b_n : n \in \mathbb{N}\} \in \mathbf{B}$.

$S_{fin}(\mathbf{A}, \mathbf{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathbf{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ of finite (possibly empty) sets such that $B_n \subset A_n$ for each n and $\bigcup_n B_n \in \mathbf{B}$.

There are infinitely long games corresponding to these selection principles.

$G_1(\mathbf{A}, \mathbf{B})$ denotes the game for two players, ONE and TWO, who play a round for each positive integer n. In the nth round ONE chooses a set A_n from $\mathbf{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins the play $(A_1, b_1, \ldots, A_n, b_n, \ldots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathbf{B}$; otherwise ONE wins.

$G_{fin}(\mathbf{A}, \mathbf{B})$ denotes the game where in the nth round ONE chooses a set A_n from $\mathbf{A}$ and TWO responds by choosing a finite (possibly empty) set $B_n \subset A_n$. TWO wins the play $(A_1, B_1, \ldots, A_n, B_n, \ldots)$ if $\bigcup_n B_n \in \mathbf{B}$ otherwise ONE wins.

In [18] the following selection principles were introduced and studied (see also [29], [6]).

The symbol $\alpha_i(\mathbf{A}, \mathbf{B})$, $i = 1, 2, 3, 4$, denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of infinite elements from $\mathbf{A}$ there is an element $B \in \mathbf{B}$ such that:

$\alpha_1(\mathbf{A}, \mathbf{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;

$\alpha_2(\mathbf{A}, \mathbf{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_3(\mathbf{A}, \mathbf{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_4(\mathbf{A}, \mathbf{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

We now recall some classes of open covers which we will use throughout the paper. If Δ is a collection of subsets of the topological space X then an open cover $\mathcal{U}$ of X is called a Δ-*cover* if X does not belong to $\mathcal{U}$ and every member of Δ is contained in a member of $\mathcal{U}$. $\mathcal{O}_\Delta$ will denote the family of all Δ-covers. When Δ is $\mathbb{F}(X)(\mathbb{K}(X))$ the Δ covers are called ω-*covers* (k-*covers*), and we use the symbols Ω ($\mathbf{K}$) to denote the set of ω-covers (k-covers). A countable Δ-cover $\mathcal{U}$ is called *groupable* (or Δ-*groupble*) if it can be represented as a countable union of finite families $\mathcal{U}_n, n \in \mathbb{N}$, where $\mathcal{U}_m \cap \mathcal{U}_n = \phi$ whenever $m \neq n$ such that for each $D \in \Delta$, for all but finitely many n there is $U \in \mathcal{U}_n$ such that $D \subset U$. The symbol $\mathcal{O}_\Delta^{gp}$ is used to denote the set of Δ-groupable covers. For $\Delta = \mathbb{F}(X)$ (= $\mathbb{K}(X)$), groupable Δ-covers are called *groupable* ω-*covers* (*groupable k-covers*), and for Δ the family of singletons we use the term groupability only (see [19]). The sets of groupable, ω-groupable and k-groupable covers will be denoted by $\mathcal{O}^{gp}$, Ω^{gp} and $\mathbf{K}^{\mathbf{gp}}$ respectively. A Δ-cover $\mathcal{U}$ is called a γ_Δ-*cover* if for each $D \in \Delta$ the set $\{U \in \mathcal{U} : D \nsubseteq U\}$ is finite. The symbol Γ_Δ denotes the set of all γ_Δ-covers. $\gamma_{\mathbb{F}(X)}$-covers ($\gamma_{\mathbb{K}(X)}$-covers) are called γ-covers (γ_k-covers) and the set of such covers is denoted by Γ (Γ_k).

Throughout I will stand for a proper ideal of $\mathbb{N}$.

We now introduce the following definitions.

Let Δ be a family of subsets of a space X. A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be:

1. An I-γ-*cover* (I-γ_Δ-cover) if for each $x \in X$ (for each $D \in \Delta$) the set $\{n \in \mathbb{N} : x \notin U_n\}$ ($\{n \in \mathbb{N} : D \nsubseteq U_n\}$) belongs to I.

2. An I-*groupable cover* (I-Δ-*groupable cover*) if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{V}_n, n \in \mathbb{N}$, such that for each $x \in X$ (for each $D \subset \Delta$) the set $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\}$ ($\{n \in \mathbb{N} : D \nsubseteq V$ for every $V \in \mathcal{V}_n\}$) belongs to I.

We denote the set of all I-γ-covers (I-γ_Δ-covers, I-γ_k-covers, I-groupable covers, I-Δ-groupable covers, I-ω-groupable covers, I-k-covers) by I-Γ (I-Γ_Δ, I-Γ_k, I-$\mathcal{O}^{gp}$, I-$\mathcal{O}_\Delta^{gp}$, I-Ω^{gp}, I-$\mathbf{K}^{\mathbf{gp}}$). For the ideal of all finite subsets of $\mathbb{N}$, $I = I_{fin}$ we get the standard notions of γ-covers (γ_Δ-covers, groupable covers, Δ-groupable covers etc) and for $I = I_d$, the ideal of zero density sets, we get the statistical variants of these notions [5], [6].

Lemma 1. [5] *An open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a topological space X is an I-γ-cover of X if and only if for each finite set $F \subset X$ the set $\{n \in \mathbb{N} : F \nsubseteq U_n\} \in I$.*

Evidently every γ-cover of X is also an I-γ-cover of X when I is admissible but the converse is not true.

Example 1. [5] Let $X = \mathbb{R}\backslash\{0\}$, the set of all nonzero real numbers with the usual topology. Let I be an admissible ideal of $\mathbb{N}$ and $I \neq I_{fin}$ the ideal of all finite subsets of $\mathbb{N}$. Then I must contain an infinite set $A = \{m_1 < m_2 < m_3, \dots\}$ (say). Without any loss of generality take $m_1 > 1$. Put $U_1 = (-\infty, 0)$ and $U_{m_i} = (0, m_i)$ for all $i \geq 1$ and $U_i = X$ for all i, $i \neq m_j$ for any j. Consider the open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X. Then $\mathcal{U}$ is an $I-\gamma$-cover of X. If $x \in (-\infty, 0)$ then $x \notin U_{m_i}$ for all $i \in \mathbb{N}$ but the set $\{m_i : i \in \mathbb{N}\} \in I$. If $x \in [0, \infty)$ then we can always choose a $m_j > x$ and so x does not belong to U_1 and at most finite number of U_{M_i}s and the result follows from the fact that I is admissible.

On the other hand $\mathcal{U}$ is clearly not a γ-cover of X since for any $x \in (-\infty, 0)$, x does not belong to infinite number of members of $\mathcal{U}$.

It is well known that any infinite subset of a γ-cover is also a γ-cover but definitely this is not true for I-γ-covers. However we can prove the following.

We call a subset $\mathcal{V}$ of a cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X I-dense in $\mathcal{U}$ if the set $M = \{m_1 < m_2 < m_3 < \dots\}$ of indices of elements from $\mathcal{V}$ belongs to $F(I)$ and further if $f : \mathbb{N} \to M$ be the bijection given by $f(i) = m_i$ then $f(A) \in I$ if and only if $A \in I$.

Lemma 2. [5] *An I-dense subset of an I-γ-cover of X is also an I-γ-cover of X.*

Lemma 3. [5] *Let $(\mathcal{U}_n : n \in \mathbb{N})$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a sequence of (countable) I-γ-covers of X. Then $(\mathcal{V}_n : n \in \mathbb{N})$, defined by*

$$\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m} : m \in \mathbb{N}\}\backslash\{\phi\}$$

is also a sequence of I-γ-covers of X.

2.2. **Applications of ideals in function spaces.** Let X be a Tychonoff space and let $\mathbf{C(X)}$ be the set of all continuous real valued functions on X. As usual $\mathbf{C_p(X)}$ and $\mathbf{C_k(X)}$ will denote the space $\mathbf{C(X)}$ with pointwise topology and the compact-open topology, respectively. Recall that the basic open neighborhoods of $f \in \mathbf{C(X)}$ in these two topologies are respectively given by:

$$W(f;F;\epsilon) = \{g \in \mathbf{C_p(X)} : |\mathbf{g(x)} - \mathbf{f(x)}| < \epsilon \forall \mathbf{x} \in \mathbf{F}\}, \mathbf{F} \in \mathbb{F}(\mathbf{X}), \epsilon > \mathbf{0}.$$

and $W(f;K;\epsilon) = \{g \in \mathbf{C_k(X)} : |\mathbf{g(x)} - \mathbf{f(x)}| < \epsilon \forall \mathbf{x} \in \mathbf{K}\}, \mathbf{K} \in \mathbb{K}(\mathbf{X}), \epsilon > \mathbf{0}.$

$\underline{0}$ will denote the function which takes the constant value 0 everywhere on X.

For a space X and a point $u \in X$, let $\Omega_u = \{A \subset X : x \in \overline{A}\backslash A\}$ and let $I - \Sigma_u$ denote the set of all sequences in X which are I-convergent to u.

Definition 1. X will be called *ideal Strictly Fréchet-Uryshon space* (I-SFU space in short) if $S_1(\Omega_u, I-\Sigma_u)$ holds for each $u \in X$.

For more details of Fréchet-Uryshon spaces and related properties one can see [13].

Theorem 1. *For a Tychonoff space X, $\mathbf{C_p}(\mathbf{X})\,(\mathbf{C_k}(\mathbf{X}))$ is an I-SFU space if and only if X satisfies $S_1(\Omega, I-\Gamma)$ $(S_1(\mathbf{K}, \mathbf{I}-\mathbf{\Gamma_k}))$.*

Proof. We present the detailed proof for $\mathbf{C_p}(\mathbf{X})$ only. First assume that $\mathbf{C_p}(\mathbf{X})$ is an I-SFU space. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of ω-covers of X. Fix $n \in \mathbb{N}$. for each finite subset F of X, there exists $U_F \in \mathcal{U}_n$ with $F \subset U_F$. Since X is Tychonoff, there exists a $f_{F,U_F} \in \mathbf{C}(\mathbf{X})$ such that $f_{F,U_F}(F) = 0$, $f_{F,U_F}(X \backslash U_F) = 1$. Let $A_n = \{f_{F,U_F} : F \in \mathbb{F}(X), F \subset U_F \in \mathcal{U}_n\}$. Then $(A_n : n \in \mathbb{N})$ is a sequence of subsets of $\mathbf{C_p}(\mathbf{X})$ for which $\underline{0} \in \overline{A}_n \forall n$. By our assumption we can then choose $f_{F_n,U_{F_n}} \in A_n$ for each n such that $(f_{F_n,U_{F_n}} : n \in \mathbb{N})$ is I-convergent to $\underline{0}$. Hence for any neighborhood W of $\underline{0}$,

$$\{n \in \mathbb{N} : f_{F_n,U_{F_n}} \notin W\} \in I.$$

Now let M be a finite subset of X and let $G = W(\underline{0}; M; \epsilon)$ for given $\epsilon > 0$. Then $\{n \in \mathbb{N} : f_{F_n,U_{F_n}} \notin G\} \in I$. But since $\{n \in \mathbb{N} : M \not\subseteq U_{F_n}\} \subset \{n \in \mathbb{N} : f_{F_n,U_{F_n}} \notin G\}$, so $\{n \in \mathbb{N} : M \not\subseteq U_{F_n}\} \in I$ and this implies that X satisfies $S_1(\Omega, I-\Gamma)$.

Conversely assume that X satisfies $S_1(\Omega, I-\Gamma)$. Let $\{A_n : n \in \mathbb{N}\} \subset \mathbf{C_p}(\mathbf{X})$ and $\underline{0} \in \overline{A_n}$ $\forall$ n. Fix $n \in \mathbb{N}$. For any finite subset F of X, choose $f_{F,n} \in A_n \cap W(\underline{0}; F; \frac{1}{n})$. By the continuity of $f_{F,n}$, choose an open neighborhood $U_{F,n}$ of F such that $f_{F,n}(U_{F,n}) \subset (-\frac{1}{n}, \frac{1}{n})$. Then $\mathcal{U}_n = \{U_{F,n} : F \in \mathbb{F}(X)\}$ is an ω-cover of X. Now by our assumption, we can choose $U_{F_n,n} \in \mathcal{U}_n$ for each n so that $(U_{F_n,n} : n \in \mathbb{N})$ forms an I-γ-cover of X. For any finite subset E of X then $\{n \in \mathbb{N} : E \not\subseteq U_{F_n,n}\} \in I$. Now consider a basic neighborhood of $\underline{0}$ of the form $W(\underline{0}; E; \frac{1}{m})$. Note that

$$f_{F_n,n}(E) \subset (-\frac{1}{n}, \frac{1}{n}) \subset (-\frac{1}{m}, \frac{1}{m}) \ \forall \ n > m.$$

Thus

$$\{n \in \mathbb{N} : f_{F_n,n} \notin W(\underline{0}; E; \frac{1}{m})\} \subset \{n \in \mathbb{N} : E \not\subseteq U_{F_n,n}\} \cup \{1, 2, 3, \ldots, m\}.$$

Since I is admissible, the set on the right hand side belongs to I and so also then the set on the left hand side. Thus $(f_{F_n,n} : n \in \mathbb{N})$ is I-convergent to $\underline{0}$. □

For the next result, recall that a space X is said to have *countable strong fan tightness* if for each $x \in X$, $S_1(\Omega_x, \Omega_x)$ holds. X has the *Reznichenko property* if for each $x \in X$, each countable $A \in \Omega_x$ is a member of Ω_x^{gp}, where Ω_x^{gp} is the set of all countable $A \in \Omega_x$ such that there are finite, pairwise disjoint sets $B_n \subset A, n \in \mathbb{N}$ such that for each neighborhood U of x meets all but finite number of B_ns (see [19], [6]).

In a similar way we can introduce the following definition.

Definition 2. A space X has *I-Reznichenko property* at a point $x \in X$ if each countable $A \in \Omega_x$ is a member of I-Ω_x^{gp} where I-Ω_x^{gp} is the set of all countable $A \in \Omega_x$ such that there are finite pairwise disjoint sets $B_n \subset A, n \in \mathbb{N}$ such that for each neighborhood U of x, $\{n \in \mathbb{N} : U \cap B_n = \phi\} \in I$. If X has I-Reznichenko property at each point $x \in X$ then we say that X has the *I-Reznichenko property.*

Theorem 2. *For a Tychonoff k-Lindelöff space X (every k-cover of X has a countable k-cover) the following are equivalent.*

(1) *ONE has no winning strategy in the game $G_1(\mathbf{K}, I\text{-}\mathbf{K^{gp}})$ on X,*

(2) $\mathbf{C_k(X)}$ *has countable strong fan tightness and the I-Reznichenko property.*

Proof. (1) $\Rightarrow$ (2) that $\mathbf{C_k(X)}$ has countable strong fan tightness follows from the fact that (1) implies X has $S_1(\mathbf{K}, \mathbf{I}-\mathbf{K^{gp}})$ and so also $S_1(\mathbf{K}, \mathbf{K})$ in view of [15, Theorem 2.2].

To prove that $\mathbf{C_k(X)}$ has I-Reznichenko property, take $A \subset \mathbf{C_k(X)}$ with $\underline{0} \in \overline{A}$. Without any loss of generality we can assume that A is countable. Construct a game of ONE and TWO in $G_1(\mathbf{K}, \mathbf{I} - \mathbf{K^{gp}})$

$$\mathcal{U}_1, U_{k_1}; \mathcal{U}_2, U_{k_2}; \mathcal{U}_3, U_{k_3}; \dots\dots\dots$$

exactly as in Theorem 4.3 [6]. Since this game is lost by ONE, $\{U_{k_n} : n \in \mathbb{N}\}$ is an I-groupable k-cover of X. Hence there exists $n_1 < n_2 < n_3 < \dots < n_k < \dots$ in $\mathbb{N}$ such that $\mathcal{W}_k = \{U_{k_j} : n_k \leq j < n_{k+1}\}, k \in \mathbb{N}$ are pairwise disjoint and for each compact subset K of X, $\{k \in \mathbb{N} : K \nsubseteq W$ for each $W \in \mathcal{W}_k\} \in I$. Consider disjoint finite sets $B_k = \{f_{k_j} : n_k \leq j < n_{k+1}\}$ of A obtained during the play such that $A = \cup_{k \in \mathbb{N}} B_k$ (see Theorem 4.3 [6]). Choose a basic neighborhood $W(\underline{0}, K, \frac{1}{n})$ of $\underline{0}$ in $\mathbf{C_k(X)}$. Choose $k \in \mathbb{N}, k > n$ and $W_k \in \mathcal{W}_k$ such that $K \subset W_k$. Since every member of $F(I)$ is infinite (I being admissible) it is possible. The corresponding $f_k \in B_k$ will satisfy the property that $f_k(W_k) \subset (-\frac{1}{k}, \frac{1}{k})$. Now

$$\{k \in \mathbb{N} : f_k \in W(\underline{0}, K, \frac{1}{n})\} \in F(I)$$

and so

$$\{k \in \mathbb{N} : B_k \cap W(\underline{0}K, \frac{1}{n}) \neq \phi\} \in F(I).$$

Hence (2) holds.

(2) $\Rightarrow$ (1) This proof is also exactly parallel to the proof of the corresponding part of Theorem 4.3 [6] in view of the fact that finite intersection of members of $F(I)$ is in $F(I)$. □

2.3. **Applications to hyperspaces.** We will now present certain selection principles for hyperspaces using ideals and ideal convergence which automatically extends some results of [6]. First we recall the following concepts.

2^X will denote the collection of all closed subsets of a Hausdorff topological space X. For $A \subset X$ and any family $\mathcal{A}$ of subsets of X we define

$$A^c = X \backslash A, \mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$

$$A^- = \{F \in 2^X : F \cap A \neq \phi\}$$

$$A^+ = \{F \in 2^X : F \subset A\}.$$

Let $\Delta \subset 2^X$ and let Δ be closed under finite unions and contains all singletons.

The *upper* Δ*-topology* denoted by τ_{Δ^+}, is the topology with the base

$$\{(D^c)^+ : D \in \Delta\} \cup \{\mathbf{CL(X)}\}.$$

When $\Delta = \mathbb{CL}(X)$, the resulting topology is known as the *upper Vietoris topology* τ_{V^+} and for $\Delta = \mathbb{K}(X)$, the resulting topology is known as upper Fell topology τ_{F^+}.

The *lower Vietoris topology* τ_{V^-} is generated by the subbase $\{U^- : U \subset X \text{is open}\}$. The Δ-topology τ_Δ is defined as $\tau_{\delta^+} \vee \tau_{V^-}$ and so has basic sets of the form

$$(D^c)^+ \cap (\bigcap_{i \leq m} V_i^-), D \in \Delta, V_1, \ldots\ldots, V_m \text{open in } X.$$

Similarly the *Vietoris topology* and *Fell topology* are defined as $\tau_V = \tau_{V^+} \vee \tau_{V^-}$, $\tau_F = \tau_{F^+} \vee \tau_{V^-}$ by [8], [10].

Following [6], for a subspace Y of X and two families of covers $\mathbf{A}, \mathbf{B}$ of Y by open sets (in X), we say that Y is an $S_1(\mathbf{A}, \mathbf{B})$ set in X if $S_1(\mathbf{A}, \mathbf{B})$ holds for Y.

Theorem 3. *For a space X, $\Delta \subset 2^X$ and for any open set $Y \subset X$ we have*

(1) *if $(2^X, \tau_\Delta)$ satisfies $S_1(\Omega_E, I - \Sigma_E)$ for each $E \in 2^X$ then Y satisfies $S_1(O_\Delta, I - \Gamma_\Delta)$.*

(2) (*i*) *$(2^X, \tau_{\Delta^+})$ satisfies $S_1(\Omega_E, I - \Gamma_E)$ for each $E \in 2^X$ if and only if*

(*ii*) *Y is an $S_1(O_\Delta, I - \Gamma_\Delta)$ set in X.*

Proof. (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of Δ-covers of Y. Then $(\mathcal{A}_n = \mathcal{U}_n^c : n \in \mathbb{N})$ is in 2^X. Then as in [8] we can show that $Y^c \in \bigcap_{n \in \mathbb{N}} cl_{\tau_\Delta}(\mathcal{A}_n)$.

Now since $(2^X, \tau_\Delta)$ satisfies $S_1(\Omega_{Y^c}, I - \Sigma_{Y^c})$, we can choose $A_n \in \mathcal{A}_n$ for each n so that the resulting sequence $(\mathcal{A}_n : n \in \mathbb{N})$ is I-convergent to Y^c in τ_Δ. Take $U_n = A_n^c, \forall n \in \mathbb{N}$. See that for any $D \in \Delta, D \subset Y$,

$$\{n \in \mathbb{N} : U_n^c \notin (D^c)^+\} \in I.$$

But since $\{n \in \mathbb{N} : D \subset U_n\} \subset \{n \in \mathbb{N} : U_n^c \notin (D^c)^+\}$, $\{n \in \mathbb{N} : D \subset U_n\} \in I$ and hence $\{U_n : n \in \mathbb{N}\}$ is an $I - \gamma_\Delta$-cover of Y.

(2) $(i) \Rightarrow (ii)$ can be proved by following similar arguments given above.

$(ii) \Rightarrow (i)$ Take a sequence $(\mathcal{A}_n : n \in \mathbb{N})$ from 2^X and let $E \in \bigcap_{n\in\mathbb{N}} cl_{\tau_{\delta+}}(\mathcal{A}_n)$. Now $\mathcal{U}_n = \mathcal{A}_n^c$ form a Δ-cover of E^c where each member of $\mathcal{U}_n$ is obviously open in X. (ii) implies that for each $n \in \mathbb{N}$, we can choose $A_n^c \in \mathcal{A}_n^c$ so that $\{A_n^c : n \in \mathbb{N}\}$ is an $I - \gamma_\Delta$-cover of E^c. Clearly then the sequence $(A_n : n \in \mathbb{N})$ is an I-convergent sequence to E in the topology τ_{Δ^+}. □

Definition 3. A space X is said to have *I-Δ-Gerlits-Nagy property* if it satisfies $S_1(\mathcal{O}_\Delta, I - \mathcal{O}_\Delta^{gp})$.

Theorem 4. *For a space X and a collection $\Delta \subset 2^X$, the following hold.*

(1) *if $(2^X, \tau_\Delta)$ satisfies $S_1(\Omega_E, I - \Omega_E^{gp})$ for each $E \in 2^X$ then each open subset Y of X has the I-Δ-Gerlits-Nagy property.*

(2) *For each $E \in 2^X$, $(2^X, \tau_{\Delta^+})$ satisfies $S_1(\Omega_E, I - \Omega_E^{gp})$ if and only if each open set $Y \subset X$ has the I-Δ-Gerlits-Nagy property in X.*

Proof. (1) Let $Y \subset X$ be open and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of Δ-covers of Y. Then as in the previous theorem, $(\mathcal{U}_n^c : n \in \mathbb{N}) \subset 2^X$ and $Y^c \in cl_{\tau_\Delta}(\mathcal{U}_n^c), \forall n \in \mathbb{N}$. By our assumption we can find $U_n \in \mathcal{U}_n$ such that $(U_n^c : n \in \mathbb{N})$ can be expressed as $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ where $\mathcal{B}_n$ are finite, pairwise disjoint and for each τ_Δ-neighborhood W of Y^c, $\{n \in \mathbb{N} : W \cap \mathcal{B}_n = \phi\} \in I$. Let $\mathcal{V}_n = \mathcal{B}_n^c$ for each $n \in \mathbb{N}$. Then $\mathcal{V}_n \subset \mathcal{U}_n$ and $\mathcal{V}_n$s are all finite and pairwise disjoint. Let $D \in \delta, D \subset Y$. Since $(D^c)^+$ is a τ_Δ-neighborhood of Y^c so $\{n \in \mathbb{N} : (D^c)^+ \cap \mathcal{B}_n = \phi\} \in I$. Since $\{n \in \mathbb{N} : D \nsubseteq V$ for each $V \in \mathcal{V}_n\} \subset \{n \in \mathbb{N} : (D^c)^+ \cap \mathcal{B}_n = \phi\}$ so the set on the left hand side also belongs to I and this shows that Y has the I-Δ-Gerlits-Nagy property.

(2) The only if part is parallel to the proof of (1).

To prove the converse, consider a sequence $(\mathcal{A}_n : n \in \mathbb{N})$ in 2^X and let $E \in \bigcap_{n\in\mathbb{N}} cl_{\tau_{\Delta^+}}(\mathcal{A}_n)$ for $E \in 2^X$. Clearly $\mathcal{U}_n = \mathcal{A}_n^c$ forms an open Δ-cover of E^c for each n. Since E^c is open and so satisfies the I-Δ-Gerlits-Nagy property, so we can choose $U_n \in \mathcal{U}_n, n \in \mathbb{N}$ such that $\mathcal{V} = \{U_n : n \in \mathbb{N}\}$ is the union of finite, pairwise disjoint families $\mathcal{V}_n$ and for each $D \in \Delta$, $D \subset E^c, \{n \in \mathbb{N} :$ there is $V \in \mathcal{V}_n$with $D \subset V\} \in F(I)$.

Clearly then if $\mathcal{B}_n = \mathcal{V}_n^c$ then if $(D^c)^+$ be a τ_{Δ^+}-neighborhood of E then $D \subset E^c$ and so $\{n \in \mathbb{N} : (D^c)^+ \cap \mathcal{B}_n \neq \phi\} \supset \{n \in \mathbb{N} : \text{there is } V \in \mathcal{V}_n \text{with } D \subset V\}$. Then the set on the left hand side also belongs to $F(I)$. Now setting $A_n = U_n^c \in \mathcal{A}_n$ we can readily observe that $(2^X, \tau_{\Delta^+})$ satisfies $S_1(\Omega_E, I - \Omega_E^{gp})$. □

Theorem 5. *For a space X, a collection $\Delta \subset 2^X$ and $E \in 2^X$, following statements are equivalent.*

(1) $(2^X, \tau_{\Delta^+})$ *satisfies* $\alpha_2(I - \Sigma_E, \Sigma_E)$;
(2) $(2^X, \tau_{\Delta^+})$ *satisfies* $\alpha_3(I - \Sigma_E, \Sigma_E)$;
(3) $(2^X, \tau_{\Delta^+})$ *satisfies* $\alpha_4(I - \Sigma_E, \Sigma_E)$;
(4) $(2^X, \tau_{\Delta^+})$ *satisfies* $S_1(I - \Sigma_E, \Sigma_E)$.

The proof is parallel to [9; Theorem 1], [6; Theorem 5.3], straightforward and so is omitted.

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