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PERIODIC ORIENTATION REVERSING HOMEOMORPHISMS OF $\mathbb{S}^2$ WITH AN INVARIANT PSEUDO-CIRCLE

J. P. BOROŃSKI

Abstract. We show that for any $k > 1$ there is a $2k$-periodic orientation reversing homeomorphism of $\mathbb{S}^2$ with an invariant pseudo-circle.

1. Introduction

In 1951, M. L. Cartwright and J. E. Littlewood [12] proved that any orientation preserving planar homeomorphism must have a fixed point in an invariant nonseparating plane continuum. The motivation for their work emerged from the study of the van der Pol equation

$$\ddot{x} + k(x^2 - 1)\dot{x} + \omega^2 x = bk \cos 2\pi t$$

that led to the invariant set whose boundary was not locally connected, and possibly contained indecomposable continua. The counterpart of the Cartwright-Littlewood theorem for orientation reversing homeomorphisms was proved by Harold Bell [5] in 1978. Since then, continua invariant under surface homeomorphism received a lot of attention in the mathematical literature; see, for example, [1], [2], [3], [4], [19], [20]. Naturally, the question arises as what continua can occur as invariant sets of surface homeomorphisms. In 1966, V. A. Pliss [22] showed that any nonseparating plane continuum is the maximal bounded closed set invariant under a transformation F, where F is a solution of certain dissipative system of differential equations (see also [1]). Michael Handel [17] constructed an area preserving C^∞ diffeomorphism of the plane with a minimal set (i.e., invariant and closed set that contains no other set with this

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property) that is a pseudo-circle. On the other hand, no example of an orientation reversing surface homeomorphism with an invariant pseudo-circle can be found in the mathematical literature. Moreover, it is known that complicated continua cannot occur as the fixed point sets for such homeomorphisms. For example, any component of the fixed point set of an orientation reversing homeomorphism of the 2-sphere $\mathbb{S}^2$ must be either a circle, an arc, or a point [13]. In 1991, answering a question of Krystyna Kuperberg motivated by her results in [19] and [20], David P. Bellamy and Wayne Lewis [7] constructed an orientation reversing homeomorphism of the plane with an invariant pseudo-arc. The purpose of the present note is to show that the construction due to Bellamy and Lewis can be used to exhibit a family of orientation reversing homeomorphisms of $\mathbb{S}^2$ with an invariant pseudo-circle.

Theorem 1.1. *For any $k \geq 1$, there is a $2k$-periodic orientation reversing homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ with an invariant pseudo-circle .*

The pseudo-circle, first described by R. H. Bing [9] in 1951, is a planar hereditarily indecomposable circle-like continuum which separates the plane into two components. It is topologically unique [15] and not homogeneous [14], [23]. In [18], Jo W. Heath showed that the pseudo-circle admits a 2-fold cover onto itself. Her example yields a construction of such a d-fold cover for any $d \geq 2$ (see [6] and [16] for related results). In [10], the author showed that if $f : \mathcal{C} \to \mathcal{C}$ is a self-map of the pseudo-circle $\mathcal{C}$ embedded essentially into an annulus $\mathbb{A}$ (i.e., $\mathcal{C}$ separates the two components of the boundary of $\mathbb{A}$) and f extends to a map $F : \mathbb{A} \to \mathbb{A}$ of degree d, then f has $|d-1|$ fixed points. The main result of the present paper provides, for any $d > 0$, a new class of self-maps of $\mathcal{C}$ that extend to degree d self-maps of $\mathbb{A}$. The extensions of these maps to self-maps of $\mathbb{A}$ interchange the two boundary components of $\mathbb{A}$, unlike those presented by Heath. For example, it is enough to compose any homeomorphism given by Theorem 1.1 with a d-fold covering map of Heath.

2. Preliminaries

Let Y be a space and $f : Y \to Y$ be a homeomorphism. Denote by $\mathrm{id}_Y : Y \to Y$ the identity map on Y. f is a *k-periodic* homeomorphism if $f^k = \mathrm{id}_Y$ and $f^i \neq \mathrm{id}_Y$ for any positive integer $i < k$. A map is *2-to-1* if the preimage of each point in the image has exactly two points. A map is *reduced* if no proper subcontinuum of the image has a connected preimage. A map is *confluent* if for each continuum X in the image, each component of the preimage of X maps onto X. We shall use a result of Heath from [18], where she showed that if f is a reduced, confluent, 2-to-1 map from a continuum X onto Y, then X is hereditarily indecomposable if and only if Y is hereditarily indecomposable. The reader is referred to [8] and [9] for definitions of *chain*, *circular chain*, *crooked (circular) chain*.

First, we shall recall from [7] the main elements of the construction of an orientation reversing planar homeomorphism with an invariant pseudo-arc, that inspired the results of the present paper. The construction is outlined in the following steps.

(1) There exists an essential degree one embedding of the pseudo-circle $\mathcal{C}$ in the Möbius band. To obtain the embedding, it is enough to form an appropriate sequence of circular chains in the Möbius band $\mathbb{M} = \mathbb{S}^1 \times [0,1]/\approx$, with the quotient map $q : (\theta, r) \approx (\theta + \pi, 1 - r)$. To obtain the initial chain, one considers the three line segments $\bar{ab}$, $\bar{bc}$, $\bar{cd}$ joining points $a = (0, \frac{1}{4}), b = (\pi, \frac{1}{3}), c = (0, \frac{1}{2})$, and $d = a = (\pi, \frac{3}{4})$. The union of those three is a simple closed curve embedded with degree 1 into $\mathbb{M}$. This spanning curve is the centerline for a circular chain $\mathcal{D}_1$ of disks of small diameter, where each two adjacent disks intersect each other in a disk. The union of $\mathcal{D}_1$ forms another Möbius band inside $\mathbb{M}$. The subsequent circular chains $\mathcal{D}_n$ are obtained by an inductive procedure which assures that $\mathcal{D}_n$ is crooked inside $\mathcal{D}_{n-1}$ and inscribed with degree 1 into it, so that the intersection $\bigcap_{n\in\mathbb{N}} \mathcal{D}_n$ is the pseudo-circle. Additionally, the spanning simple closed curve of each $\mathcal{D}_n$ is invariant in $\mathbb{S}^1 \times [0,1]$ under $(\theta, r) \mapsto (\theta + \pi, 1 - r)$.

(2) The universal covering of $\mathbb{M}$ is $\tilde{\mathbb{M}} = \mathbb{R} \times [0,1]$ with a covering map $\tau(x, y) = (x (\operatorname{mod} \pi), 1 - y)$.

(3) The deck transformation $\tilde{f}$ determined by the generator of the fundamental group of $\mathbb{M}$ produces an orientation reversing homeomorphism of $\tilde{\mathbb{M}}$, which is invariant on the fiber of the pseudo-circle $\tau^{-1}(\mathcal{C})$, and interchanges the two boundary components of $\tilde{\mathbb{M}}$. See Figure 1.

(4) The two-point compactification K of $\tilde{\mathbb{M}}$ is a disk, with the two-point compactification of the lift of the pseudo-circle homeomorphic to the pseudo-arc. The desired homeomorphism is obtained by the fact that $\tilde{f}$ extends to a homeomorphism of K with the pseudo-arc invariant.

Note that $\tau^{-1}(\mathcal{C})$ in the above construction is connected, as it is the common boundary of $\tau^{-1}(K_1)$ and $\tau^{-1}(K_2)$, where K_1 and K_2 are the two components of $\mathbb{A} \setminus \mathcal{C}$ (see [24, Theorem 1(iii)$'$]).

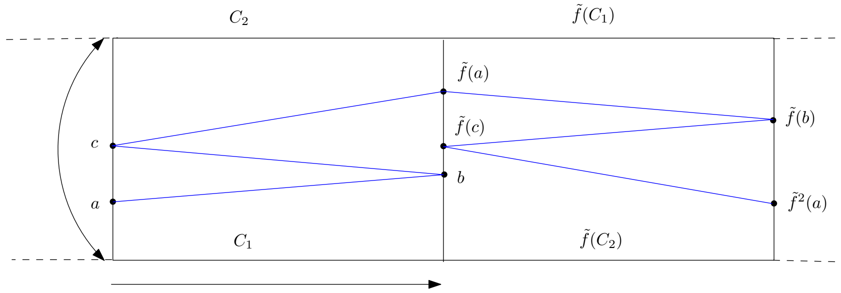

FIGURE 1. Action of the deck transformation $\tilde{f}$ on $\tilde{\mathbb{M}}$ with a lift of the spanning line of $\mathcal{D}_1$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. First, we will describe an example of such a homeomorphism for $k = 1$. Consider the pseudo-circle $\mathcal{C}$ embedded with degree 1, as described in [7], into the Möbius band $\mathbb{M} = \mathbb{S}^1 \times [0,1]/\approx$, with the quotient map $q : (\theta, r) \approx (\theta + \pi, 1 - r)$. Then the annulus $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ provides a natural 2-fold cover of $\mathbb{M}$ with the quotient map q as the covering map. Let $\mathcal{K} = q^{-1}(\mathcal{C})$.

CLAIM 1. $\mathcal{K}$ is a separating plane continuum.

Proof of Claim 1. Notice that $\mathcal{K}$ is a continuum as $\tau^{-1}(\mathcal{C})$, the lift of $\mathcal{C}$ to the universal cover $(\tilde{\mathbb{M}}, \tau)$, is connected. Indeed, $\mathcal{K}$ must have either one or two components. If $\mathcal{K}$ had two components, then $\tau^{-1}(\mathcal{C})$ would need to have two components in $\tilde{\mathbb{M}}$, as $\tau^{-1}(\mathcal{C})$ covers also $\mathcal{K}$. However, $\tau^{-1}(\mathcal{C})$ is connected, as noted earlier.

$\mathcal{K}$ is a separating plane continuum, as $\mathcal{K}$ separates the two boundary components of $\mathbb{A}$, by the fact that $\tau^{-1}(\mathcal{C})$ separates the two boundary components of $\tilde{\mathbb{M}}$.

CLAIM 2. $\mathcal{K}$ is circle-like.

Proof of Claim 2. Let $\mathcal{U}^\epsilon$ be a finite ϵ-cover of $\mathcal{C}$ by open sets, with the nerve $N(\mathcal{U}^\epsilon)$ of U^ϵ homeomorphic to $\mathbb{S}^1$. Let also $\{U^\epsilon_n : n \leq d\}$ consist of disjoint homeomorphic copies of U^ϵ in $q^{-1}(U^\epsilon)$. We shall show that, for sufficiently small ϵ, $\mathcal{V}^\epsilon = \{U^\epsilon_n : n \leq d, U^\epsilon \in \mathcal{U}^\epsilon\}$ is an ϵ-cover of $\mathcal{K}$ and $N(\mathcal{V}^\epsilon)$ is homeomorphic to $\mathbb{S}^1$.

It is clear that $\mathcal{V}^\epsilon$ is a cover of $\mathcal{K}$. Choosing the distance between two points (θ, r) and (θ', r') to be given by $d((\theta, r), (\theta', r') = |\theta - \theta'| + |r - r'|$ for both $\mathbb{M}$ and $\mathbb{A}$, q is a local isometry. Therefore, for sufficiently small

ϵ, $\mathcal{V}^\epsilon$ is an ϵ-cover of $\mathcal{K}$. To see that $N(\mathcal{V}^\epsilon)$ is topologically the circle, we shall show that any vertex of $N(\mathcal{V}^\epsilon)$ is of degree 2.

Let $\tilde{V}_0$ be an element of $\mathcal{V}^\epsilon$, and set $U_0 = q(\tilde{V}_0)$. By definition, $U_0 \in \mathcal{U}^\epsilon$, and since $N(\mathcal{U}^\epsilon)$ is topologically $\mathbb{S}^1$, there are exactly two elements, say U_{-1} and U_1 of $\mathcal{U}^\epsilon$ such that $U_{-1} \cap U_0 \neq \emptyset \neq U_1 \cap U_0$. Therefore, if $\tilde{W} \in \mathcal{V}^\epsilon$, then $\tilde{W} \cap \tilde{V} \neq \emptyset$ if and only if $q(\tilde{W}) \in \{U_{-1}, U_1\}$.

Because $U_0 \cap U_{-1} \neq \emptyset$, there is $\tilde{V}_{-1}$, a component of $q^{-1}(U_{-1})$, such that $\tilde{V}_{-1} \cap \tilde{V}_0 \neq \emptyset$. Similarly for U_1. Consequently, each vertex of $N(\mathcal{V}^\epsilon)$ has degree at least 2.

Suppose there is another component $\tilde{Z}_{-1}$ of $q^{-1}(U_{-1})$ such that $\tilde{V}_{-1} \cap \tilde{V}_0 \neq \emptyset$. For sufficiently small ϵ, q restricted to $\tilde{Z}_{-1} \cup \tilde{V}_{-1} \cup \tilde{V}_0$ is a homeomorphism resulting in a contradiction. Consequently, each vertex of $N(\mathcal{V}^\epsilon)$ has degree at most 2. Clearly, $N(\mathcal{V}^\epsilon)$ is homeomorphic to $\mathbb{S}^1$.

CLAIM 3. $\mathcal{K}$ is hereditarily indecomposable.

Proof of Claim 3. Every subcontinuum of $\mathcal{C}$ is the pseudo-arc. For any such pseudo-arc, there is a disk D containing it that lifts to two disjoint homeomorphic copies of D in $\mathbb{A}$. Therefore, no proper subcontinuum $Y \subseteq \mathcal{C}$ has a connected preimage $q^{-1}(Y)$, and consequently q is a reduced map. Clearly, q is also a confluent map by the very definition of a covering map. Finally, it is 2-to-1, and therefore, by [18, Lemma 1], $\mathcal{K}$ is hereditarily indecomposable as $\mathcal{C}$ is.

It follows that $\mathcal{K}$ is a circle-like hereditarily indecomposable separating plane continuum, and therefore, it is the pseudo-circle (see, for example, [15]).

Now, define a homeomorphism $g : \mathbb{A} \to \mathbb{A}$ to be the deck transformation determined by the generator of the fundamental group of $\mathbb{M}$. g can be viewed as the rotation of $\mathbb{A}$ by 180° followed by the reflection that interchanges the two boundary components of $\mathbb{A}$. To obtain a homeomorphism of $\mathbb{S}^2$, identify all points that lie on the same component of the boundary of $\mathbb{A}$. Since g^2 is the identity, g is 2-periodic. See Figure 2.

To finish the proof, note that although Heath's original example was only for 2-fold cover, it is well known that the same arguments can be applied for any other n-fold cover (see, for example, [16]). Therefore, to obtain an example for any $2k > 2$, consider the annulus that is a $2k$-fold cover of $\mathbb{M}$ and apply the same arguments. □

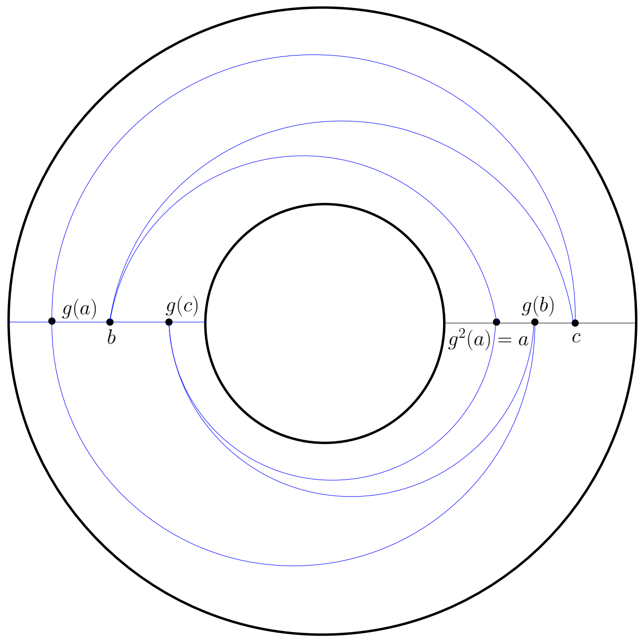

FIGURE 2. The 2-periodic homeomorphism g

4. Further Comments on Self-Maps of the Pseudo-Circle

One can see the 2-periodic homeomorphisms of the pseudo-circle as analogues of antipodal maps of $\mathbb{S}^1$. Any two fixed-point free antipodal maps of $\mathbb{S}^1$ are isotopic, as any fixed-point free homeomorphism of $\mathbb{S}^1$ must be isotopic to the identity map. Also, it is possible to formulate a version of the Borsuk-Ulam theorem [11] for self-maps of the pseudo-circle. Namely, let $a : \mathcal{C} \to \mathcal{C}$ be a fixed-point free map such that $a^2(x) = a \circ a(x) = x$ for any $x \in \mathcal{C}$. Then for any self-map of the pseudo-circle $F : \mathcal{C} \to \mathcal{C}$, such that $F(\mathcal{C}) \neq \mathcal{C}$, there is a $z \in \mathcal{C}$ for which $F(z) = F(a(z))$. This is a consequence of the fact that any self-map of the pseudo-circle that is not surjective maps the pseudo-circle onto a pseudo-arc. Then one can apply the following known property of chainable continua [21, Theorem 12.29]: for any continuum X, any chainable continuum Y, and any two maps $f, g : X \to Y$, such that either $f(X) = Y$ or $g(X) = Y$, there is a $z \in X$ such that $f(z) = g(z)$. Therefore, it seems interesting to point out that the 2-periodic homeomorphism of the pseudo-circle $a(x)$ constructed by Heath and the homeomorphism $g(x)$ exhibited in Theorem

1.1 are, in some sense, not equivalent. This is because the pseudo-circle separates $\mathbb{S}^2$ into two complementary domains U_1 and U_2. If z is a point in the pseudo-circle accessible from U_1, then its antipode $a(z)$ has the same property. On the other hand, $g(z)$, the antipode under g, is accessible from U_2, and therefore, it cannot be accessible from U_1. Otherwise, since there is no fixed point for g, there would be two points accessible from both complementary domains and the pseudo-circle would be the union of two continua meeting at those two points, contradicting its indecomposability.

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