TOPOLOGY PROCEEDINGS
Volume 40, 2012
Pages 297–301

http://topology.auburn.edu/tp/

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Sudip Kumar Acharyya and Swapan Kumar Ghosh

Electronically published on February 6, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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E-Published on February 6, 2012

A NOTE ON FUNCTIONS IN $C(X)$ WITH SUPPORT LYING ON AN IDEAL OF CLOSED SUBSETS OF X

SUDIP KUMAR ACHARYYA AND SWAPAN KUMAR GHOSH

Abstract. For any ideal $\mathcal{P}$ of closed subsets of a completely regular Hausdorff space X, we introduced $\mathcal{P}$-compact spaces in an earlier paper. In this note we show that for any infinite cardinal number θ, if $\mathcal{P}_\theta$ is the ideal of closed finally θ-compact subsets of a realcompact space X, then X is $\mathcal{P}_\theta$-compact if and only if it is finally θ-compact. This improves our earlier result, where X was a discrete space with $\theta = \omega_1$. Using this new result, we establish further that if θ is a regular cardinal number, and $\mathcal{P}$ is the ideal of all closed θ-compact subsets of a realcompact space X, then X is $\mathcal{P}$-compact if and only if it is θ-compact.

1. Introduction

Throughout, X will stand for a completely regular Hausdorff topological space. In an earlier paper of ours [1], a family $\mathcal{P}$ of closed subsets of X is called an *ideal of closed sets* if it satisfies two conditions: (i) If $A, B \in \mathcal{P}$ then $A \cup B \in \mathcal{P}$. (ii) If $A \in \mathcal{P}$ and $B \subseteq A$ with B closed in X, then $B \in \mathcal{P}$.

For any such ideal $\mathcal{P}$ of closed subsets of a space X, let $C_\mathcal{P}(X) = \{f \in C(X) : cl_X(X - Z(f)) \in \mathcal{P}\}$, where $Z(f) = \{x \in X : f(x) = 0\}$ stands for the zero-set of f.

Also we called X a $\mathcal{P}$-compact space if for any point $p \in \beta X - X$, there is an $f \in C_\mathcal{P}(X)$ such that $f^*(p) = \infty$, where $f^* : \beta X \to \mathbb{R} \cup \{\infty\}$ is the uniquely determined continuous extension of f, and βX denotes the Stone-Čech compactification of X.

2010 *Mathematics Subject Classification.* Primary: 54C40, Secondary: 46E25 .

Key words and phrases. finally θ-compact spaces, locally θ-compact spaces, $\mathcal{P}$-compact spaces, $\mathcal{P}$-stable family, realcompact spaces, regular cardinal number, supports of continuous functions, θ-compact spaces.

It is clear that every $\mathcal{P}$-compact space is realcompact. In addition, if $\mathcal{P}$ is the family of all closed subsets of a space X, then X is $\mathcal{P}$-compact if and only if it is realcompact. On the other hand if $\mathcal{P}$ is the family of all compact subsets of X, then X is $\mathcal{P}$-compact if and only if it is compact. So it is a fascinating question to ask how far the known special classes of realcompact spaces could be realized as $\mathcal{P}$-compact for suitable natural choices of the family $\mathcal{P}$. It may be mentioned in this context that in paper [1], we established the following:

Theorem 1.1. *A discrete space X is Lindelöf if and only if it is $\mathcal{L}$-compact where $\mathcal{L}$ is the ideal of all closed Lindelöf subsets of X.*

In the present note we prove that the same conclusion holds without the discreteness hypothesis. We achieve this as a special case of a more general result (see section 3) which implies that a realcompact space X is finally θ-compact if and only if it is $\mathcal{P}_\theta$-compact, where $\mathcal{P}_\theta$ is the ideal of all closed finally θ-compact subsets of X and θ is an infinite cardinal number. Using this result in the same section, we further prove that if θ is a regular cardinal number and $\mathcal{P}$ is the ideal of all closed θ-compact subsets of a realcompact space X, then X is θ-compact if and only if it is $\mathcal{P}$-compact. A particularly interesting case of this fact obtained by putting $\theta = \omega_1$ is that a space X is σ-compact when and only when it is $\mathcal{P}$-compact, where $\mathcal{P}$ is the ideal of all closed σ-compact subsets of X. It may be noted that for $\theta > \omega_1$, a θ-compact space may fail to be realcompact. For example, the space of all countable ordinals $[0, \omega_1)$ is ω_2-compact without being realcompact.

2. Terminology and a Few Technicalities

To make our note self-contained, we reproduce the following definitions and facts from our earlier paper [1] and also from a paper of R. M. Stephenson, Jr. [4].

(i) A family $\mathcal{F}$ of subsets of X is called $\mathcal{P}$-*stable* if every function in $C_{\mathcal{P}}(X)$ is bounded on some set in $\mathcal{F}$. [1, Definition 3.5]

To establish the first main result of the present note, we shall use the following characterization of $\mathcal{P}$-compact spaces.

(ii) A space X is $\mathcal{P}$-compact if and only if every $\mathcal{P}$-stable family of closed sets in X with the finite intersection property has nonvoid intersection. [1, Theorem 3.6]

(iii) For any infinite cardinal number θ, a space X is called *finally θ-compact* if each open cover of X has a subcover with cardinality $< \theta$. [4]

Thus, in this terminology, finally ω_0-compact spaces and finally ω_1-compact spaces are just compact spaces and Lindelöf spaces, respectively.

(iv) A space X is called *θ-compact* if there is a family $\{X_\alpha : \alpha \in \Lambda\}$ of compact subsets of X with $\text{card}(\Lambda) < \theta$ and $X = \cup\{X_\alpha : \alpha \in \Lambda\}$. [4]

(v) A space X is called *locally θ-compact* if, for each $x \in X$, there exists a θ-compact open neighborhood W_x of x in X. [4]

Clearly, ω_1-compact spaces are precisely the σ-compact spaces. It is easy to check that the family of all closed finally θ-compact subsets of X and also the family of all closed θ-compact subsets of X are both ideals of closed sets in X. Since a σ-compact space is Lindelöf, one might ask whether for any infinite cardinal number θ, a θ-compact space is also finally θ-compact. We do not know the answer in general, but a suitable restriction on θ entails an affirmative answer to this question. We now need the following notion.

(vi) A cardinal number θ is called *regular* if it is not the sum of fewer smaller cardinal numbers, or equivalently, if there does not exist any co-final subset of θ with cardinality $< \theta$; otherwise, θ is called a *singular cardinal number*. [4]

It is well known that, for any non-limit ordinal number α, ω_α is a regular cardinal number while $\omega_{\omega_0} = \sum_{n \in \omega_0} \omega_n$ is a singular cardinal number.

Routine arguments yield the following result.

(vii) Let θ be a regular cardinal number. Then every θ-compact space X is finally θ-compact. [4]

3. The Main Results

Theorem 3.1. *Let X be realcompact and $\mathcal{P}_\theta$ be the ideal of closed finally θ-compact subsets of X. Then X is $\mathcal{P}_\theta$-compact if and only if it is finally θ-compact.*

Proof. First, assume that X is finally θ-compact and then choose $p \in \beta X - X$. Since X is realcompact, there is an $f \in C(X)$ such that $f^*(p) = \infty$. Then $cl_X(X - Z(f))$, like any closed subset of X, is finally θ-compact. Hence, X must be $\mathcal{P}_\theta$-compact.

To prove the converse, assume that X is not finally θ-compact. Then there exists a family $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ of closed sets in X with $\cap \mathcal{B} = \emptyset$ and with the property that for any subfamily $\mathcal{B}_0$ of $\mathcal{B}$ with $\text{card}(\mathcal{B}_0) < \theta$, $\cap \mathcal{B}_0 \neq \emptyset$. Let $\mathcal{E} = \{E_\alpha : \alpha \in \Lambda^*\}$ be the family of all sets E_α such that each E_α is an intersection of $< \theta$ sets belonging to the family $\mathcal{B}$. Then $\cap \mathcal{E} = \emptyset$ and $\mathcal{E}$ has the finite intersection property. To show that

X is not $\mathcal{P}_\theta$-compact, it remains to check, in view of remark (ii), that $\mathcal{E}$ is a $\mathcal{P}_\theta$-stable family. For that purpose, choose $f \in C(X)$ such that $cl_X(X - Z(f)) \in \mathcal{P}_\theta$. Then, since $\{X - B_\alpha : \alpha \in \Lambda\}$ is an open cover of X, we can write $cl_X(X - Z(f)) \subseteq \cup_{\alpha \in \Lambda_1}(X - B_\alpha)$ for some subfamily Λ_1 of Λ with card$(\Lambda_1) < \theta$. Consequently, the set $\cap_{\alpha \in \Lambda_1} B_\alpha \subseteq Z(f)$ with $\cap_{\alpha \in \Lambda_1} B_\alpha \in \mathcal{E}$. Thus, f becomes bounded on a set in the family $\mathcal{E}$, which is therefore a $\mathcal{P}_\theta$-stable family. And that completes the proof. □

Theorem 3.2. *Let X be realcompact and the infinite cardinal number θ be regular. Suppose $\mathcal{P}$ is the ideal of all closed θ-compact subsets of X. Then X is $\mathcal{P}$-compact if and only if it is θ-compact.*

Proof. If $\theta = \omega_0$, the result is trivial. Assume that $\theta \geq \omega_1$. We observe that, by a proof analogous to the first part of the proof of Theorem 3.1, if X is θ-compact, then it is obviously $\mathcal{P}$-compact.

Conversely, let X be $\mathcal{P}$-compact. Now since θ is regular, it follows from remark (vii) that $\mathcal{P} \subseteq \mathcal{P}_\theta$, where the latter ideal is defined as in Theorem 3.1. Hence, X is $\mathcal{P}_\theta$-compact. Therefore, by Theorem 3.1, X is finally θ-compact. If possible let X be not θ-compact. Then X can't be locally θ-compact since, otherwise, for each $x \in X$, there exists a θ-compact open neighborhood W_x of x in X. But as X is finally θ-compact, the open covering $\{W_x : x \in X\}$ of X has a subcover $\{W_x : x \in X_0\}$ for some subset X_0 of X with card$(X_0) < \theta$. As θ is regular, the union of less than θ many θ-compact spaces is also θ-compact. This implies that $X = \cup_{x \in X_0} W_x$ is θ-compact. Let $X_1 = \{x \in X : x$ has no θ-compact open neighborhood in $X\}$. Since the set T of points of X at which X is locally θ-compact satisfies $X = T \cup X_1$, and since T is θ-compact, X_1 can't be compact. We can, therefore, choose a point $p \in \beta X - X$ such that $p \in cl_{\beta X} X_1$. Now by using the $\mathcal{P}$-compactness of X, we can select an $f \in C(X)$ such that $f \geq 0$, $f^*(p) = \infty$ and $cl_X(X - Z(f))$ is θ-compact. Let $A = f^{-1}([0,1])$, then $p \notin cl_{\beta X} A$. Since $\theta \geq \omega_1$, every cozero subset of a θ-compact space is θ-compact. Therefore, $X - Z(f)$ is θ-compact. Since $f^{-1}([1,\infty)) \subseteq X - Z(f)$, it follows that X_1 is disjoint with $f^{-1}([1,\infty))$. Hence, $X_1 \subseteq f^{-1}([0,1]) = A$. Consequently, $p \in cl_{\beta X} A$. This contradiction proves our theorem. □

Acknowledgments. The authors are grateful to Professor Alan Dow for suggesting the technique for the proof of Theorem 3.2. The authors also express their gratitude to the referee for helping a lot in the substantial improvement of the original version of this paper.

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(Acharyya & Ghosh) Department of Pure Mathematics; University of Calcutta; 35, Ballygunge Circular Road; Kolkata-700019, West Bengal, India

E-mail address, Acharyya: sdpacharyya@gmail.com

E-mail address, Ghosh: swapan12345@yahoo.co.in