THE CENTER AND EXTENDED CENTER
OF THE MAXIMAL GROUPS IN
THE SMALLEST IDEAL OF $\beta\mathbb{N}$

by

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Abstract. A good deal is known about the maximal groups in the smallest ideal $K(\beta N)$ of the compact right topological semigroup $(\beta N, +)$. For example they are pairwise isomorphic and highly non-commutative – they contain a copy of the free group on $2^n$ generators. If $q$ is an idempotent in $K(\beta N)$, then $Z + q$ is contained in the center of the maximal group $q + \beta N + q$. We do not know whether that center is equal to $Z + q$. In this paper we investigate the center of $q + \beta N + q$ and the extended center consisting of all elements of $\beta N$ that commute with every element of $q + \beta N + q$. This extended center trivially includes all idempotents $r$ of $\beta N$ such that $q \leq r$, as well as elements of the form $n + r$ for such $r$ and for $n \in Z$. We show, for example, that if those are the only elements of the extended center, then there are no nontrivial continuous homomorphisms from $\beta N$ to $\beta N \setminus N$. This would answer a long standing open question. We include several other open questions.

1. Introduction

Addition on the set $N$ of positive integers extends to the Stone-Čech compactification $\beta N$ of $N$ making $(\beta N, +)$ a right topological semigroup (meaning that for each $p \in \beta N$, the function $\rho_p : \beta N \to \beta N$ is continuous, where $\rho_p(q) = q + p$) with $N$ contained in its topological center (meaning that for each $n \in N$, the function $\lambda_n : \beta N \to \beta N$ is continuous,}

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where $\lambda_n(q) = n + q$. As with any compact Hausdorff right topological semigroup, $(\beta\mathbb{N}, +)$ has a smallest two sided ideal

$$
K(\beta\mathbb{N}) = \bigcup\{L : L \text{ is a minimal left ideal of } \beta\mathbb{N}\} = \bigcup\{R : R \text{ is a minimal right ideal of } \beta\mathbb{N}\}.
$$

Any left ideal contains a minimal left ideal, which is closed, and any right ideal contains a minimal right ideal. If $L$ is a minimal left ideal and $R$ is a minimal right ideal, then $L \cap R$ is a group and $L \cap R = q + \beta\mathbb{N} + q$ where $q$ is the unique idempotent in $L \cap R$. Any two such groups are isomorphic. If $q$ and $r$ are idempotents in the same minimal right ideal, then the restriction of $\rho_r$ to $q + \beta\mathbb{N} + q$ is an isomorphism and a homeomorphism onto $r + \beta\mathbb{N} + r$.

The facts just mentioned about $K(\beta\mathbb{N})$ are true in any compact Hausdorff right topological semigroup. Many additional facts are known about $K(\beta\mathbb{N})$ that do not hold in all such semigroups. We know, for example, that there are $2^c$ minimal right ideals and $2^c$ minimal left ideals and the maximal groups in $K(\beta\mathbb{N})$ each contain a copy of the free group on $2^c$ generators. We also know that the center of $\beta\mathbb{Z}$ is $\mathbb{Z}$ so if $q$ is an idempotent in $K(\beta\mathbb{N})$, then $Z + q$ is contained in the center of $q + \beta\mathbb{N} + q$. We do not know whether the center of $q + \beta\mathbb{N} + q$ is equal to $Z + q$. It is this question which is the primary focus of this paper.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. Given $A \subseteq \mathbb{N}$, the closure $\overline{A}$ of $A$ in $\beta\mathbb{N}$ is $\{p \in \beta\mathbb{N} : A \in p\}$ and $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets of $\beta\mathbb{N}$. See [4] for an elementary introduction to the topology and the algebraic structure of $\beta S$ where $S$ is an infinite discrete semigroup, as well as for proofs of all of the facts mentioned in the paragraphs above. (The original references for these facts are [1], [2], [3], [5], [6], and [7].)

**Definition 1.1.** Let $q$ be an idempotent in $K(\beta\mathbb{N})$. $G_q = q + \beta\mathbb{N} + q$ and $D_q = \{u \in \mathbb{N}^* : (\forall v \in G_q) (u + v = v + u)\}$.

Of course, the center $Z(G_q) = D_q \cap G_q$. We call $D_q$ the *extended center* of $G_q$.

**Definition 1.2.**

1. $I = \bigcap_{n=1}^\infty \overline{n\mathbb{N}}$.
2. $H = \bigcap_{n=1}^\infty \overline{2n\mathbb{N}}$.
3. For $A \subseteq \beta\mathbb{N}$, $E(A) = \{q \in \beta\mathbb{N} : q + q = q\}$.

In section 2 of this paper we present some basic results, including the fact that for any $q \in E(\beta\mathbb{N})$, $D_q \subseteq Z + I$.

In section 3 we investigate more deeply the structure of $D_q$, establishing the fact that either there is no nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, or there is a member of $D_q \cap I$ which is not an
idempotent. We also include in this section a proof that $D_q$ contains a
decreasing sequence of idempotents of order type $(\omega + 1)^*$, that is, the
reverse of $\omega + 1$.

Section 4 consists primarily of a derivation of the fact that if the center
of $G_q$ is not trivial, then $G_q$ contains a copy of $\mathbb{Z} \times \mathbb{Z}$.

2. Basic Facts about the Extended Center

We begin by observing that the only elements in the extended center
that are not in the center lie outside of the smallest ideal.

**Theorem 2.1.** Let $q \in E(K(\beta N))$. Then $Z(G_q) = D_q \cap K(\beta N)$.

**Proof.** Trivially $Z(G_q) \subseteq D_q \cap K(\beta N)$. For the reverse inclusion, let
$x \in D_q \cap K(\beta N)$. Since $q \in K(\beta N)$, $\beta N + q$ is a minimal left ideal
and $q + \beta N$ is a minimal right ideal, so $G_q = (\beta N + q) \cap (q + \beta N)$. Thus, if
$x \notin G_q$, then either $x \notin \beta N + q$ or $x \notin q + \beta N$. So either $x$ and $q$
are in different minimal left ideals of $\beta N$ or $x$ and $q$ are in different minimal
right ideals of $\beta N$. In either case, $x + q \neq q + x$. \hfill \Box

The idempotents of $\beta N$ are partially ordered by the relation $\leq$, defined
by $e \leq f$ if and only if $e = e + f = f + e$. By [4, Theorem 2.9], $e$ is minimal
with respect to this order if and only if $e \in K(\beta N)$. Further,
given any non minimal idempotent $e$ in $\beta N$, by [4, Theorem 1.60], there
is a minimal idempotent $q \in K(\beta N)$ with $q \leq e$. So the following lemma
shows that for at least some $q \in K(\beta N)$, $D_q \neq Z(G_q)$.

**Lemma 2.2.** Let $q \in E(K(\beta N))$. Then $\{e \in E(\beta N) : q \leq e\} = E(D_q)$.

**Proof.** Let $e \in E(\beta N)$ such that $q \leq e$ and let $x \in G_q$. Then $e + x = e + q + x = q + x = x + q + e = x + e$ and so $e \in D_q$.

Conversely, let $e \in E(D_q)$. Since $e + q = q + e$, $e + q$ is an idempotent
in the same minimal right ideal and the same minimal left ideal as $q$. So
$e + q = q + e = q$ and $q \leq e$. \hfill \Box

The remainder of this section will be devoted to a proof, as a consequence
of a more general theorem, that $D_q \subseteq Z + I$.

It is well known, and routine to verify, that each member of $N$ has
a unique factorial representation, that is, a representation of the form
$\sum_{n \in H} a_n \cdot n!$ where $H$ is a finite nonempty subset of $N$ and for each
$n \in H$, $a_n \in \{1, 2, \ldots, n\}$.

**Definition 2.3.** Define $d : \mathbb{N} \to \times_{n=1}^\infty \{0, 1, \ldots, n\}$ by $y = \sum_{n=1}^\infty d(y)(n) \cdot n!$ for each $y \in \mathbb{N}$. For $y \in \mathbb{N}$, let $\text{supp}_y(y) = \{n \in \mathbb{N} : d(y)(n) \neq 0\}$ and
let $c(y) = |\text{supp}_y(y)|$. Let $d : \beta N \to \times_{n=1}^\infty \{0, 1, \ldots, n\}$ and $c : \beta N \to \beta N$
be the continuous extensions of $d$ and $c$, respectively.
Lemma 2.4. Let \( x \in \mathbb{N}^+ \). If \( x \notin \mathbb{Z} + I \), then

1. \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) \neq 0 \} \) is infinite,
2. \( \{ n \in \mathbb{N} : d(x)(n) < n \} \) is infinite, and
3. \( \{ n \in \mathbb{N} : \text{either} \ 0 < \tilde{d}(x)(n) < n \ \text{or both} \ \tilde{d}(x)(n) = n \ \text{and} \ \tilde{d}(x)(n+1) = 0 \} \) is infinite.

Proof. (1) Suppose that \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) \neq 0 \} \) is finite and let \( k = \max\{ n \in \mathbb{N} : \tilde{d}(x)(n) \neq 0 \} \). Let \( m = \sum_{n=1}^{k} d(x)(n) \cdot n! \). We claim that \( x \in m + I \). To see this, let \( l = k \). To see that \( x \in m + N ! \), pick \( A \in x \) such that \( \tilde{d}(x)[A] \subseteq \times_{n=1}^{l} \pi_{n}^{-1}\{ [\tilde{d}(x)(n)] \} \) and let \( y \in A \). Pick \( j > l \) such that \( j! > y \). Then

\[
y - m = \sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} d(x)(n) \cdot n! - \sum_{n=1}^{k} \tilde{d}(x)(n) \cdot n!
= \sum_{n=l+1}^{j} d(y)(n) \cdot n!
\]
since \( \tilde{d}(x)(n) \cdot n! = 0 \) for \( n > k \).

(2) Suppose that \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) < n \} \) is finite and pick \( k \in N \) such that for all \( n > k \), \( \tilde{d}(x)(n) = n \). Let \( m = 1 + \sum_{n=1}^{k} (n - \tilde{d}(x)(n)) \cdot n! \). We claim that \( x \in -m + I \). To see this, let \( l = k \). To see that \( x \in m + N ! \), pick \( A \in x \) such that \( \tilde{d}(x)[A] \subseteq \times_{n=1}^{l} \pi_{n}^{-1}\{ [\tilde{d}(x)(n)] \} \) and let \( y \in A \). Pick \( j > l \) such that \( j! > y \). Then \( m + y - 1 = \sum_{n=1}^{k} (n - \tilde{d}(x)(n)) \cdot n! + \sum_{n=1}^{l} \tilde{d}(x)(n) \cdot n! + \sum_{n=l+1}^{j} d(y)(n) \cdot n! = \sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} n \cdot n! \) and so \( (l+1)! \) divides \( m + y \).

(3) Assume that \( \{ n \in \mathbb{N} : 0 < \tilde{d}(x)(n) < n \} \) is finite. Pick \( k \) such that for all \( n > k \), \( \tilde{d}(x)(n) \in \{0, n\} \). Then by (1) and (2), both \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) = 0 \} \) and \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) = n \} \) are infinite, and consequently \( \{ n \in \mathbb{N} : \tilde{d}(x)(n) = n \} \) and \( \tilde{d}(x)(n+1) = 0 \) is infinite.

Lemma 2.5. Let \( x \in \mathbb{N}^+ \) and let \( q, r \in I \). If \( q + x + r \in \mathbb{Z} + I \), then \( x \in \mathbb{Z} + I \).

Proof. Assume that \( m \in \mathbb{Z}, z \in I \), and \( q + x + r = m + z \). Given any \( n \in \mathbb{N}, \{ k \in \mathbb{N} : k \equiv m (\text{mod} \ n!) \} \in q + x + r \) and \( \{ k \in \mathbb{N} : k \equiv 0 (\text{mod} \ n!) \} \in q \cap r, \) so \( \{ k \in \mathbb{N} : k \equiv m (\text{mod} \ n!) \} \subseteq x. \)

Lemma 2.6. Let \( q \in E(K(\beta\mathbb{N})) \) and let \( x \in Z(G_q) \). Then \( x \in \mathbb{Z} + I \).

Proof. Suppose that \( x \notin \mathbb{Z} + I \) and let

\[
E = \{ n \in \mathbb{N} : \text{either} 0 < \tilde{d}(x)(n) < n \ \text{or both} \ \tilde{d}(x)(n) = n \ \text{and} \ \tilde{d}(x)(n+1) = 0 \}.
\]

Then, by Lemma 2.4(3), \( E \) is infinite, so pick \( p \in \mathbb{N}^+ \) such that \( \{ n! : n \in E \} \in p \). We shall show that
Let $A$.
Assume that $\beta$ then have that $\alpha(c(x + p + q)) = \alpha(c(x + q + p)) = \alpha(c(x + q)) + 1 = \alpha(c(x)) + 1 \neq \alpha(c(x)) = \alpha(c(q + p + x)) = \alpha(c(q + p + q + x))$.

To verify (a), it suffices that $\beta \circ \rho_{p+q}$ and $\rho_1 \circ \beta \circ \rho_q$ agree on $\mathbb{N}$, so let $y \in \mathbb{N}$ be given. To see that $\beta(y + p + q) = \beta(y + q) + 1$, it suffices that $\beta \circ \lambda_y \circ \rho_q$ is constantly equal to $\beta(y + q) + 1$ on $\{n! : n \in \mathbb{N} \text{ and } n > \max\supp(f(y))\}$, so let $n \in \mathbb{N}$ such that $n > \max\supp(f(y))$. To see that $\beta(y + n! + q) = \beta(y + q) + 1$, it suffices that $\beta \circ \lambda_{y+n!}$ and $\rho_1 \circ \beta \circ \lambda_y$ agree on $\mathbb{N}(n+1)!$, so let $z \in \mathbb{N}(n+1)!$. Then $\supp_f(y + n! + z) = \supp_f(y) \cup \{n! \} \cup \supp_f(z)$ and $\supp_f(y + z) = \supp_f(y) \cup \supp_f(z)$.

To verify (b), it suffices that $\beta \circ \rho_{p+x}$ and $\beta \circ \rho_x$ agree on $\mathbb{N}$, so let $y \in \mathbb{N}$ and pick $a \in E$ such that $a > \max\supp_f(y)$. To see that $\beta(y + p + x) = \beta(y + x)$, it suffices that $\beta \circ \lambda_y \circ \rho_x$ is constantly equal to $\beta(y + x)$ on $\{n! : n \in \mathbb{E} \text{ and } n > a + 2\}$ so let $b \in \mathbb{E}$ such that $b > a + 2$. Pick $A \subseteq x$ such that $\bar{d}([A]) \subseteq \bigcap_{n=1}^{b+1} \pi_n^{-1}([\bar{d}(x)(n)])$. To see that $\beta(y + b! + x) = \beta(y + x)$, it suffices that $\beta \circ \lambda_{b!+x}$ agrees with $\beta \circ \lambda_y$ on $A$, so let $z \in A$. We need to show that $\beta(y + b! + z) = \beta(y + z)$.

Pick $l > b + 1$ such that $l! > z$. Then $z = \sum_{n=b+2}^{b+1} \bar{d}(x)(n) \cdot n! + \sum_{n=1}^{b+1} \bar{d}(x)(n) \cdot n!$. Since $b > a + 2 > a > \max\supp_f(y)$, there is no carrying beyond position $a + 1$ when the factorial representations of $z$ and $y$ are added. (Either $\bar{d}(x)(a) < n$, in which case there is no carrying beyond position $a$, or $\bar{d}(x)(a) = n$ and $\bar{d}(x)(a + 1) = 0$.) Thus, $z + y = \sum_{n=b+2}^{l} \bar{d}(x)(n) \cdot n! + \sum_{n=a+2}^{b+1} \bar{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} \bar{d}(z + y)(n) \cdot n!$.

Assume first that $0 < \bar{d}(x)(b) < n$. Then
\[
y + b! + z = \sum_{n=b+2}^{l} \bar{d}(x)(n) \cdot n! + \bar{d}(x)(b + 1) \cdot (b + 1)! + \bar{d}(x)(b + 1) \cdot b! + \sum_{n=a+2}^{b+1} \bar{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} \bar{d}(z + y)(n) \cdot n!;
\]
sup $f(y + b! + z)$ is constantly equal to $\alpha(c(x + p + q)) = \alpha(c(x + q + p)) = \alpha(c(x + q)) + 1 = \alpha(c(x)) + 1 \neq \alpha(c(x)) = \alpha(c(q + p + x)) = \alpha(c(q + p + q + x))$.

Now assume that $\bar{d}(x)(b) = n$ and $\bar{d}(x)(b + 1) = 0$. Then $y + b! + z = \sum_{n=b+2}^{l} \bar{d}(z)(n) \cdot n! + (b + 1)! + \sum_{n=a+2}^{b+1} \bar{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} \bar{d}(z + y)(n) \cdot n!$;
so sup $f(y + b! + z) = (\sup f(y + z) \setminus \{b\}) \cup \{b + 1\}$.

\[\square\]

**Theorem 2.7.** Let $u, v, x \in \beta \mathbb{N}$. If $x$ commutes with every member of $u + \beta \mathbb{N} + v$, then either $x \in \mathbb{N}$ or $x \in \mathbb{Z} + I$.

**Proof.** Assume that $x \notin \mathbb{N}$. Pick a minimal right ideal $R$ and a minimal left ideal $L$ of $\beta \mathbb{N}$ such that $R \subseteq u + \beta \mathbb{N}$ and $L \subseteq \beta \mathbb{N} + v$. Let $q$ be the identity of $R \cap L$. Then $q + \beta \mathbb{N} + q \subseteq u + \beta \mathbb{N} + v$ and so $x$ commutes with $\beta \mathbb{N} + v$.\[\square\]
every member of $q + \beta \mathbb{N} + q$, and consequently so does $q + x + q$. (Let $p \in q + \beta \mathbb{N} + q$. Then $p + q = q + p = p$ so $q + x + q + p = q + x + p + q = q + p + x + q = p + q + x + q$.) By Lemma 2.6, $q + x + q \in \mathbb{Z} + I$ and so by Lemma 2.5, $x \in \mathbb{Z} + I$. \hfill \qed

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8. Let $q \in E(K(\beta \mathbb{N}))$. Then $D_q \subseteq \mathbb{Z} + I$.

3. THE STRUCTURE OF THE EXTENDED CENTER

Section 1.7 of [4] has a large number of results whose hypothesis asserts the existence of a minimal left ideal with an idempotent. The following theorem puts all of those results at our disposal.

Theorem 3.1. Let $q \in E(K(\beta \mathbb{N}))$. Then $D_q$ is a semigroup and $D_q \cap G_q$ is both a minimal left ideal of $D_q$ with an idempotent and a minimal right ideal of $D_q$.

Proof. Trivially, $D_q$ is a semigroup. To see that $D_q \cap G_q$ is an ideal of $D_q$, let $x \in D_q \cap G_q$ and let $y \in D_q$. Then $y + x = y + q + x = q + y + x = q + y + x + q \in G_q$ and $x + y = x + q + y = x + y + q = q + x + y + q \in G_q$. Since $D_q \cap G_q = Z(G_q)$, $D_q \cap G_q$ is a group and is therefore a minimal left ideal and a minimal right ideal and has an idempotent. \hfill \qed

Among the consequences of the existence of a minimal left ideal in a semigroup is the fact that the smallest ideal exists.

Corollary 3.2. Let $q \in E(K(\beta \mathbb{N}))$. Then $D_q \cap G_q = K(D_q)$.

Proof. Since $K(D_q)$ is the union of all of the minimal left ideals of $D_q$, by Theorem 3.1, we have that $D_q \cap G_q \subseteq K(D_q)$. Since $D_q \cap G_q$ is a two sided ideal of $D_q$, we have that $K(D_q) \subseteq D_q \cap G_q$. \hfill \qed

Lemma 3.3. Let $q \in E(K(\beta \mathbb{N}))$ and let $x \in D_q$. There is some $y \in D_q \cap G_q$ such that $y + x = x + y = q$.

Proof. Since $D_q$ contains a minimal left ideal with an idempotent, by [4, Corollary 1.47 and Theorem 1.56], we have that $D_q + x$ contains a minimal left ideal of $D_q$ with an idempotent, and this idempotent is in $K(D_q) = D_q \cap G_q$. Since $q$ is the only idempotent in $G_q$, $q \in D_q + x$. Pick $w \in D_q$ such that $q = w + x$. Let $y = w + q$. Then $y \in D_q \cap G_q$. Also, $y + x = q + w + q + x = q + w + x + q = q + q + q = q$. Since $y \in D_q$, we also have that $x + y = q$. \hfill \qed

Theorem 3.4. Let $q \in E(K(\beta \mathbb{N}))$ and let $x \in D_q$. Then $G_q = x + G_q + x = x + G_q = G_q + x$. 


Proof. Pick, by Lemma 3.3, some $y \in D_q \cap G_q$ such that $y + x = x + y = q$. Since $x \in D_q$, we have that $x + G_q = G_q + x$. We shall show that $G_q \subseteq x + G_q + x \subseteq x + G_q \subseteq G_q$. Let $w \in G_q$. Then $w = q + w + q = x + y + w + y + x \in x + G_q + x$.

To see that $x + G_q + x \subseteq x + G_q$, let $w \in G_q$ and pick $z \in G_q$ such that $z + y = w$. Then $x + w + x = x + z + y + x = x + z + q = x + z$.

To see that $x + G_q \subseteq G_q$, let $w \in G_q$, and pick $z \in G_q$ such that $y + z = w$. Then $x + w = x + y + z = q + z = z$. \qed

Corollary 3.5. Let $q \in E(K(\beta \mathbb{N}))$. For any distinct $x, y \in D_q$, $x \in \beta \mathbb{N} + y$ or $y \in \beta \mathbb{N} + x$.

Proof. By Theorem 3.4, $q \in (G_q + x) \cap (G_q + y)$. So our claim follows from [4, Corollary 6.21]. \qed

Corollary 3.6. Let $q \in E(K(\beta \mathbb{N}))$. If $M$ is a $G_\delta$ subset of $\mathbb{N}^*$, then $D_q \cap M$ is nowhere dense in $M$. In particular, $D_q \cap I$ is nowhere dense in $I$ and $D_q \cap \mathbb{H}$ is nowhere dense in $\mathbb{H}$.

Proof. We first observe that $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ contains a dense open subset $U$ of $\mathbb{N}^*$. This follows from the fact that, if $p \in \mathbb{N}^*$ and $B \in p$, we can choose a sequence $(x_n)_{n=1}^\infty$ contained in $B$ such that $x_{n+1} - x_n > n$ for every $n \in \mathbb{N}$. So, if $A = \{x_n : n \in \mathbb{N}\}, A \subseteq B$ and, by [4, Exercise 4.1.7], $A \cap (\mathbb{N}^* + \mathbb{N}^*) = \emptyset$.

If there exists an element $x \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, then, by Corollary 3.5, for any element $y \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, $x \in \mathbb{N} + y$ or $y \in \mathbb{N} + x$. So $D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)) = \mathbb{Z} + x$. Since $\mathbb{Z} + x$ is countable, $\mathbb{Z} + x$ is nowhere dense in $\mathbb{N}^*$ [4, Corollary 3.37]. Put $V = U$ if no such element $x$ exists; otherwise, put $V = U \setminus cl(\mathbb{Z} + x)$. Then $V$ is a dense open subset of $\mathbb{N}^*$ and $V \cap D_q = \emptyset$.

It follows from [4, Theorem 3.36] that $int_{\mathbb{N}^*}(M)$ is dense in $M$. So $V \cap M$ is a dense open subset of $M$ disjoint from $D_q$. \qed

Given $q \in E(K(\beta \mathbb{N}))$, we know $(\mathbb{Z} + E(D_q)) \subseteq D_q$ and we know, by Lemma 2.2, that $E(D_q) = \{e \in E(\beta \mathbb{N}) : q \leq e\}$. So the only things that we know are in $D_q \cap I$ are the idempotents above $q$. It is a longstanding open problem as to whether there are any nontrivial continuous homomorphisms from $\beta \mathbb{N}$ to $\mathbb{N}^*$. The following theorem connects our lack of knowledge about these two issues.

Theorem 3.7. If for all $q \in E(K(\beta \mathbb{N}))$, $D_q \cap I \subseteq E(\beta \mathbb{N})$, then there is no nontrivial continuous homomorphism from $\beta \mathbb{N}$ to $\mathbb{N}^*$.

Proof. Let $q \in E(K(\beta \mathbb{N}))$ and suppose that $D_q \cap I \subseteq E(\beta \mathbb{N})$ and that there is a nontrivial continuous homomorphism from $\beta \mathbb{N}$ to $\mathbb{N}^*$. By [4, Corollary 10.20], pick $e \in E(\beta \mathbb{N})$ and $p \neq e$ such that $p + p = p + e = $
Let $q \in E(K(\beta \mathbb{N}))$ such that $q \leq e$. Then $p + q = p + e + q = e + q = q + e = q + e + p = q + p$. We claim that $p \in D_q$, so let $w \in G_q$. Then $p + w = p + q + w = q + w = w + q = w + q + p = w + p$. Thus, $p \in D_q$. Since $p + q = q$, we also have that $p \in I$. But $p$ is not an idempotent.

We do not know whether there are any maximal idempotents in $\beta \mathbb{N}$ or even whether there are maximal idempotents in $K(\beta \mathbb{N})$, so, as far as we know, it is possible that, for some $q \in E(K(\beta \mathbb{N}))$, the extended center $D_q$ of $G_q$ is equal to the center of $G_q$. We shall show in Theorem 3.9 that for many $q \in E(K(\beta \mathbb{N}))$, $D_q \cap \text{cl} K(\beta \mathbb{N})$ contains an infinite decreasing chain of idempotents. As a corollary, we obtain the fact that $\text{cl} K(\beta \mathbb{N})$ contains a decreasing sequence of idempotents of reverse order type $\omega + 1$. (It was previously known that it contains such a sequence of reverse order type $\omega$.)

**Lemma 3.8.** Let $R$ be a minimal right ideal of $\beta \mathbb{N}$. There is an injective sequence $\langle q_n \rangle_{n=1}^\infty$ of idempotents in $R$ such that if $p$ is an accumulation point of $\langle q_n \rangle_{n=1}^\infty$, then $p \notin \mathbb{Z}^* + \mathbb{Z}^*$. In particular, any accumulation point of $\langle q_n \rangle_{n=1}^\infty$ is right cancelable in $\beta \mathbb{Z}$.

**Proof.** Pick an injective sequence $\langle v_n \rangle_{n=1}^\infty$ in

$$\{2^n : n \in \mathbb{N}\}^* = \{2^n : n \in \mathbb{N}\} \setminus \mathbb{N}.$$  

We claim that $(\beta \mathbb{N} + v_n) \cap (\beta \mathbb{N} + v_m) = \emptyset$ when $n \neq m$. To see this, define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max (\text{supp}(n))$, where $\text{supp}(n)$ is the binary support of $n$, that is, $n = \sum_{t \in \text{supp}(n)} 2^t$. Let $\bar{\phi} : \beta \mathbb{N} \to \beta \omega$ be the continuous extension of $\phi$. By [4, Exercise 3.4.1], $\bar{\phi}$ is injective on $\{2^n : n \in \mathbb{N}\}$ and by [4, Lemma 6.8], for each $n \in \mathbb{N}$, $\bar{\phi}[\beta \mathbb{N} + v_n] = \{\bar{\phi}(v_n)\}$, so the claim is established.

For each $n \in \mathbb{N}$ choose an idempotent $q_n \in R \cap (\beta \mathbb{N} + v_n)$ and note that $q_n \neq q_m$ if $n \neq m$. Let $p$ be an accumulation point of $\langle q_n \rangle_{n=1}^\infty$ and suppose that $p = x + y$ for some $x, y \in \mathbb{Z}^*$. By [4, Exercise 4.3.5], $y \in \mathbb{Z}^*$. Note that there is at most one $n \in \mathbb{Z}$ such that $n + y \notin \mathbb{N}$. (If $n < m$ and $2^k > m - n$, then $(-n + 2^k \mathbb{N}) \cap (-m + 2^k \mathbb{N}) = \emptyset$.) Let $X = \{n \in \mathbb{Z} : n + y \notin \mathbb{N}\}$. Then $X \in \mathbb{Z}$. If $n \neq m$, we have that $\bar{\phi}(q_n) = \bar{\phi}(v_n) \neq \bar{\phi}(v_m) = \bar{\phi}(q_m)$, so there are at most three values of $n \in \mathbb{N}$ for which

$$\bar{\phi}(q_n) \in \{\bar{\phi}(y) - 1, \bar{\phi}(y), \bar{\phi}(y) + 1\}.$$  

Let $M = \{n \in \mathbb{N} : \bar{\phi}(q_n) \notin \{\bar{\phi}(y) - 1, \bar{\phi}(y), \bar{\phi}(y) + 1\}\}$. Then

$$p \in \text{cl} \{q_n : n \in M\} \cap \text{cl}(X + y);$$

so, by [4, Theorem 3.40], either there is some $n \in X$ such that $n + y \in \text{cl} \{q_n : n \in M\}$ or there is some $n \in M$ such that $q_n \in \text{cl}(X + y) = X + y$. 

Suppose first that we have \( n \in X \) such that \( n + y \in \text{cl}\{q_n : n \in M\} \).
By [4, Lemma 6.8], \( \{q_n : n \in M\} \subseteq \mathbb{H} \), so \( n + y \in \mathbb{H} \), contradicting the fact that \( n \in X \).

Now assume that we have \( n \in M \) such that \( q_n \in \overline{X} + y \) and pick \( z \in \overline{X} \) such that \( q_n = z + y \). Then \( \phi(q_n) \notin \{\phi(y) - 1, \phi(y) + 1\} \), so pick \( A \in \phi(q_n) \) such that \( N \setminus A \subseteq \phi(y) - 1 \), \( N \setminus A \subseteq \phi(y) \), and \( N \setminus A \subseteq \phi(y) + 1 \).

Pick \( B \in z \) such that \( \phi(B + y) \subseteq \overline{A} \) and pick \( k \in B \). Then \( \phi(k + y) \in \overline{A} \), so pick \( C \in y \) such that \( \phi(k + C) \subseteq \overline{A} \). Since \( N \setminus A \subseteq \phi(y) - 1 \), \( N \setminus A \subseteq \phi(y) \), \( N \setminus A \subseteq \phi(y) + 1 \), pick \( D \in y \) such that \( \phi(D) - 1 \subseteq N \setminus A \), \( \phi(D) \subseteq N \setminus A \), and \( \phi(D) + 1 \subseteq N \setminus A \).

Pick \( r \in C \cap D \) such that \( r > k \). Then \( \phi(k + r) = \phi(r) - 1, \phi(k + r) = \phi(r) \), or \( \phi(k + r) = \phi(r) + 1 \). Since \( \phi(k + r) \in A \), this says that \( \phi(r) - 1 \in A, \phi(r) \in A \), or \( \phi(r) + 1 \in A \), a contradiction.

The “in particular” conclusion follows from [4, Theorem 8.18]. \( \square \)

**Theorem 3.9.** Let \( R \) be a minimal right ideal of \( \beta \mathbb{N} \). There is a decreasing sequence \( \langle p_n \rangle_{n=1}^{\infty} \) of idempotents in \( \text{cl}K(\beta \mathbb{N}) \) such that
\[
\left| \{ q \in E(R) : \{ p_n : n \in \mathbb{N} \} \subseteq D_q \} \right| = 2^\mathcal{C}.
\]

**Proof.** Choose a sequence \( \langle q_n \rangle_{n=1}^{\infty} \) as guaranteed by Lemma 3.8 and pick an accumulation point \( x \) of this sequence. Then \( x \) is right cancelable in \( \beta \mathbb{Z} \). Since each \( q_n \) is in \( R \) and each idempotent in \( R \) is a right identity for \( R \), we have that for each \( n \in \mathbb{N} \) and each \( p \in E(R), q_n + p = p \), and consequently, for each \( p \in E(R) \), \( x + p = p \). Let \( M = \bigcap \{ C \subseteq \beta \mathbb{Z} : C \text{ is a compact subsemigroup of } \beta \mathbb{Z} \text{ and } x \in C \} \). Note that \( M \subseteq \beta \mathbb{N} \). For each \( p \in E(R), \{ z \in \beta \mathbb{N} : z + p = p \} \) is a compact subsemigroup of \( \beta \mathbb{Z} \) with \( x \) as a member, so we have that for all \( z \in M \) and all \( p \in E(R), z + p = p \).

By [4, Corollary 8.54], pick a decreasing sequence \( \langle p_n \rangle_{n=1}^{\infty} \) in \( M \) and let \( w \) be a cluster point of \( \langle p_n \rangle_{n=1}^{\infty} \). By [4, Lemma 9.22], \( w \) is right cancelable in \( \beta \mathbb{Z} \) and for each \( n \in \mathbb{N} \), \( w \in \beta \mathbb{Z} + p_n \). By [4, Theorem 6.56], \( \beta \mathbb{N} + w \) contains \( 2^\mathcal{C} \) pairwise disjoint left ideals. Let \( L \) be one of these and pick an idempotent \( q \in R \cap L \). To complete the proof, we show that for each \( n \in \mathbb{N}, q \leq p_n \) (so, by Corollary 2.2, \( \{ p_n : n \in \mathbb{N} \} \subseteq D_q \)). Let \( n \in \mathbb{N} \). Then \( L \subseteq \beta \mathbb{N} + w \subseteq \beta \mathbb{Z} + p_n \), so \( p_n + q = q \). Also, \( p_n \in M \) and \( q \in E(R) \), so \( p_n + q = q \). \( \square \)

**Corollary 3.10.** There exist decreasing chains of idempotents in \( \text{cl}K(\beta \mathbb{N}) \) of reverse order type \( \omega + 1 \).

**Proof.** Pick a minimal right ideal \( R \) of \( \beta \mathbb{N} \), pick a sequence \( \langle p_n \rangle_{n=1}^{\infty} \) as guaranteed by Theorem 3.9, and pick \( q \in E(R) \) such that
\[
\{ p_n : n \in \mathbb{N} \} \subseteq D_q.
\]
By Corollary 2.2, given $n \in \mathbb{N}$, $q \leq p_n$ and, since $p_{n+1} < p_n$, $q < p_n$. □

4. Copies of $\mathbb{Z} \times \mathbb{Z}$ in $G_q$

We know, of course, that if $q \in E(K(\beta \mathbb{N}))$, then the center of $G_q$ contains $\mathbb{Z} + q$. We show in this section that if it is not equal to $\mathbb{Z} + q$, then $G_q$ contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.

**Definition 4.1.** Let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m, x \in \mathbb{N}$.

- (a) $\text{supp}(x)$ is the binary support of $x$.
- (b) $c_{B_1}(x) = |\text{supp}(x) \cap B_1|$.
- (c) $c_{B_1,\ldots,B_k}(x) = \sum_{i=1}^k c_{B_i,x}$.
- (d) $c_{B_1,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,m}(x) \equiv c_{B_1}(x) \pmod{m}$.
- (e) $c_{B_1,\ldots,B_k,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,\ldots,B_k,m}(x) \equiv c_{B_1,\ldots,B_k}(x) \pmod{m}$.

**Lemma 4.2.** Let $u, v \in \mathbb{H}$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m \in \mathbb{N}$.

1. $\bar{c}_{B_1,m}(u + v) = \bar{c}_{B_1,m}(u) + \bar{c}_{B_1,m}(v)$.
2. If $k > 1$, then $\bar{c}_{B_1,\ldots,B_k,m}(u + v) = \bar{c}_{B_1,\ldots,B_k,m}(u) + \bar{c}_{B_1,\ldots,B_k,m}(v) + \sum_{i=1}^{k-1} \bar{c}_{B_1,\ldots,B_i,m}(u) \cdot \bar{c}_{B_{i+1},\ldots,B_k,m}(v)$.

**Proof.** (1) It suffices that $\bar{c}_{B_1,m} \circ \rho_v$ and $\rho_{\bar{c}_{B_1,m}(v)} \circ \bar{c}_{B_1,m}$ agree on $\mathbb{N}$, so let $x \in \mathbb{N}$. Let $k = \max \text{supp}(x) + 1$. It suffices to observe that $\bar{c}_{B_1,m} \circ \lambda_x$ and $\lambda_{\bar{c}_{B_1,m}(x)} \circ \bar{c}_{B_1,m}$ agree on $\mathbb{N}2^k$.

(2) Note that singletons are open in $\mathbb{Z}_m$. Pick $C \subseteq u$ such that for all $x \in C$, $\bar{c}_{B_1,\ldots,B_k,m}(x + y) = \bar{c}_{B_1,\ldots,B_k,m}(u + v)$ and for $t \in \{1, 2, \ldots, k\}$, $\bar{c}_{B_1,\ldots,B_k,m}(x) = \bar{c}_{B_1,\ldots,B_t,m}(x)$. Pick $x \in C$ and let $l = \max \text{supp}(x) + 1$. Pick $D \subseteq u$ such that for all $y \in D$, $c_{B_1,\ldots,B_k,m}(x + y) = \bar{c}_{B_1,\ldots,B_k,m}(x + v)$ and for $t \in \{1, 2, \ldots, k - 1\}$, $c_{B_1,\ldots,B_k,m}(y) = \bar{c}_{B_1,\ldots,B_k,m}(v)$. Pick $y \in D \cap \mathbb{N}2^k$. Then $\bar{c}_{B_1,\ldots,B_k,m}(x + y) = \bar{c}_{B_1,\ldots,B_k,m}(x) + \bar{c}_{B_1,\ldots,B_k,m}(y)$ + $\sum_{i=1}^{k-1} c_{B_1,\ldots,B_i,m}(x) \cdot \bar{c}_{B_{i+1},\ldots,B_k,m}(y)$. □

**Lemma 4.3.** Let $q \in E(K(\beta \mathbb{N}))$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m \in \mathbb{N}$. Then $\bar{c}_{B_1,\ldots,B_k,m}(q) = 0$.

**Proof.** This follows immediately by induction on $k$ from Lemma 4.2. □

**Lemma 4.4.** Let $q \in E(K(\beta \mathbb{N}))$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, let $m \in \mathbb{N}$, and let $u \in \mathbb{H} \cap D_q$. Then $\bar{c}_{B_1,\ldots,B_k,m}(u) = 0$. 

Proof. We show first that it suffices to show this under the additional assumption that \( \mathbb{N} \setminus \bigcup_{i=1}^{k} B_i \) is infinite. Suppose we have done this and let \( B'_k \) and \( B''_k \) be disjoint infinite subsets of \( B_k \) with \( B'_k \cup B''_k = B_k \). Note that for all \( x \in \mathbb{N} \), \( c_{B_1,\ldots,B_k,m}(x) = c_{B_1',\ldots,B_k',m}(x) + c_{B_1,\ldots,B_k''_k,m}(x) \) so \( c_{B_1,\ldots,B_k,m}(u) = c_{B_1,\ldots,B_k',m}(u) + c_{B_1,\ldots,B_k''_k,m}(u) = 0 + 0 \).

So assume that \( B_{k+1}\setminus\bigcup_{i=1}^{k} B_i \) is infinite. Pick \( p \in \{2^n : n \in (B_{k+1})^\ast\} \). Note that for all \( n \in B_{k+1} \), \( c_{B_{k+1},m}(2^n) = 1 \), and if \( t \in \{1,2,\ldots,k\} \), then \( c_{B_1,\ldots,B_k,m}(2^n) = c_{B_1,\ldots,B_{k+1},m}(2^n) = 0 \). Therefore, \( c_{B_{k+1},m}(p) = 1 \), and if \( t \in \{1,2,\ldots,k\} \), then \( c_{B_1,\ldots,B_{k+1},m}(p) = c_{B_1,\ldots,B_{k+1},m}(p) = 0 \). Since all terms of the expansions given in Lemma 4.2 except one involve \( q \) and are therefore 0, we have that \( c_{B_{k+1},m}(p+q) = 1 \), and if \( t \in \{1,2,\ldots,k\} \), then \( c_{B_1,\ldots,B_{k+1},m}(p+q) = c_{B_1,\ldots,B_{k+1},m}(p+q) = 0 \) and \( c_{B_{k+1},m}(p+q) = 1 \), and if \( t \in \{1,2,\ldots,k\} \), then \( c_{B_1,\ldots,B_{k+1},m}(p+q) = c_{B_1,\ldots,B_{k+1},m}(p+q) = 0 \).

Next note that

\[
\begin{align*}
c_{B_1,\ldots,B_{k+1},m}(q + u + p + q) & = c_{B_1,\ldots,B_{k+1},m}(u + q + p + q) + \sum_{t=1}^{k} c_{B_1,\ldots,B_{k+1},m}(q) \cdot c_{B_{t+1},\ldots,B_{k+1},m}(u + p + q) \equiv c_{B_1,\ldots,B_{k+1},m}(u + p + q) \quad \text{and} \\
c_{B_1,\ldots,B_{k+1},m}(q + p + u) & = c_{B_1,\ldots,B_{k+1},m}(q + p + u) + \sum_{t=1}^{k} c_{B_1,\ldots,B_{k+1},m}(q + p + u) \cdot c_{B_{t+1},\ldots,B_{k+1},m}(q) \equiv c_{B_1,\ldots,B_{k+1},m}(q + p + u) .
\end{align*}
\]

Since

\[
c_{B_1,\ldots,B_{k+1},m}(q + u + p + q) = c_{B_1,\ldots,B_{k+1},m}(u + q + p + q) = c_{B_1,\ldots,B_{k+1},m}(q + p + u + q) = c_{B_1,\ldots,B_{k+1},m}(q + p + q) .
\]

we, therefore, have that \( c_{B_1,\ldots,B_{k+1},m}(u + p + q) = c_{B_1,\ldots,B_{k+1},m}(q + p + u) \).

Now

\[
\begin{align*}
c_{B_1,\ldots,B_{k+1},m}(u + p + q) & = c_{B_1,\ldots,B_{k+1},m}(u) + c_{B_1,\ldots,B_{k+1},m}(p) \equiv c_{B_1,\ldots,B_{k+1},m}(u) + c_{B_1,\ldots,B_{k+1},m}(p) \quad \text{and} \\
c_{B_1,\ldots,B_{k+1},m}(q + p + u) & = c_{B_1,\ldots,B_{k+1},m}(q) + c_{B_1,\ldots,B_{k+1},m}(u) + \sum_{t=1}^{k} c_{B_1,\ldots,B_{k+1},m}(q + p + u) \cdot c_{B_{t+1},\ldots,B_{k+1},m}(u) \equiv c_{B_1,\ldots,B_{k+1},m}(u) .
\end{align*}
\]

Consequently, \( c_{B_1,\ldots,B_{k+1},m}(u) + c_{B_1,\ldots,B_{k+1},m}(u) = c_{B_1,\ldots,B_{k+1},m}(u) \), so \( c_{B_1,\ldots,B_{k+1},m}(u) = 0 \) \( \square \)

Lemma 4.5. Let \( q \in E(K(\beta\mathbb{N})) \), let \( p \in \{2^n : n \in \mathbb{N}\}^\ast \), and let \( \psi_p : \mathbb{Z} \rightarrow G_q \) be the homomorphism such that \( \psi_p(1) = q + p + q \). Then for all \( n \in \mathbb{Z} \setminus \{0\} \), \( \psi_p(n) \notin D_q \).
Proof. Pick infinite $B \subseteq \mathbb{N}$ such that $\{2^n : n \in B\} \in p$. Then for each $m \in \mathbb{N} \setminus \{1\}$, $\tilde{c}_{B,m}(q + p + 1) = 1$, so for all $n \in \mathbb{N}$ and all $m > n$, $\tilde{c}_{B,m}(\psi_p(n)) = n$, and thus $\psi_p(n) \not\in D_q$ by Lemma 4.4. Now $D_q \cap G_q$ is a group, so if $n \in \mathbb{N}$ and $\psi_p(-n) \in D_q$, so is $\psi_p(n)$.

Theorem 4.6. Let $q \in E(K(\beta\mathbb{N}))$, let $p \in \{2^n : n \in \mathbb{N}\}^*$, and let $\psi_p : \mathbb{Z} \rightarrow G_q$ be the homomorphism such that $\psi_p(1) = q + p + q$. Assume that $u \in \mathbb{Z} \cap G_q \cap D_q \setminus \{q\}$ and let $\varphi : \mathbb{Z} \rightarrow G_q$ be the homomorphism such that $\varphi(1) = u$. Define $\tau : \mathbb{Z} \times \mathbb{Z} \rightarrow G_q$ by $\tau(m,n) = \varphi(m) + \psi_p(n)$. Then $\tau$ is an injective homomorphism.

Proof. Given $(m,n)$ and $(k,l)$ in $\mathbb{Z} \times \mathbb{Z}$, one has that $\tau((m,n) + (k,l)) = \tau(m,n) + \tau(k,l)$ because $\varphi(k) \in D_q$. Now assume that $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $\tau(m,n) = q$. Then $\varphi(m) + \psi_p(n) = q$, so $\varphi(m) = \psi_p(-n)$, and thus $\psi_p(-n) \in D_q$ so that $n = 0$ by Lemma 4.5. Therefore, $\varphi(m) = q$. By Zelenyuk’s Theorem [8] (or see [4, Theorem 7.17]), $\beta\mathbb{N}$ contains no nontrivial finite groups. If one had $m \neq 0$, then $\varphi[\mathbb{Z}]$ would be a nontrivial finite group, so $m = 0$.

Corollary 4.7. Let $q \in E(K(\beta\mathbb{N}))$. If the center of $G_q$ is not equal to $\mathbb{Z} + q$, then $G_q$ contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.

Proof. Assume we have $x \in Z(G_q) \setminus (\mathbb{Z} + q)$. Then, by Lemma 2.6, $x \in \mathbb{Z} + I$, so pick $n \in \mathbb{Z}$ and $u \in I$ such that $x = n + u$. Then $u \in \mathbb{Z} \cap G_q \cap D_q \setminus \{q\}$. Pick any $p \in \{2^n : n \in \mathbb{N}\}^*$. Define $\tau$ as in Theorem 4.6. Then $\tau$ is an injective homomorphism.

We conclude by listing some of the tantalizing open questions that have arisen in the study of the center and extended center of $G_q$.

Question 4.8. (1) Let $q \in E(K(\beta\mathbb{N}))$. Does $Z(G_q) = \mathbb{Z} + q$?
(2) Let $q \in E(K(\beta\mathbb{N}))$. Is $D_q \subseteq \mathbb{Z} + E(\beta\mathbb{N})$?
(3) Does there exist $q \in E(K(\beta\mathbb{N}))$ for which $E(D_q)$ is finite?
(4) Does there exist $q \in E(K(\beta\mathbb{N}))$ for which $E(D_q)$ is uncountable?
(5) Let $q_1, q_2 \in E(K(\beta\mathbb{N}))$. Are $D_{q_1}$ and $D_{q_2}$ isomorphic?

References


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