On Convergent Sequences and Copies of $\beta N$ in the Stone Space of One Boolean Algebra

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IN THE STONE SPACE OF ONE BOOLEAN ALGEBRA

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Abstract. We consider the Stone space of one Boolean algebra constructed by Murray G. Bell (see Example 2.1 in Compact ccc nonseparable spaces of small weight, Topology Proc. 5 (1980), 11–25 (1981)), which is a compactification, $BN$, of a countable discrete space $N$.

We get a necessary condition for a set $A \subseteq N$ to be a convergent sequence in $BN$ and a necessary condition for a set $A \subseteq N$ to be such that $A$ is homeomorphic to $\beta N$.

1. Introduction

We consider the Stone space of the Boolean algebra constructed by Murray G. Bell [1, Example 2.1]. This space, $BN$, is a compactification of a countable discrete space, $N$, with ccc non-separable remainder.

In [5], it was proved that for any infinite chain $A$ of $N$, one gets $|A \setminus A| = 1$, i.e., $A$ is a convergent sequence [2, Theorem 3.9]. We also showed [2, Lemma 3.5] that if $A \subseteq N$ is a strict anti-chain (see the definition in §2), then $\overline{A}$ is homeomorphic to $\beta N$.

Here we get the following.

Theorem 3.1. If a set $A \subseteq N$ is such that $|\overline{A} \setminus A| = 1$, then $A \setminus K$ is a chain for some finite $K \subseteq A$.

Theorem 3.2. If the closure $\overline{A}$ of a set $A \subseteq N$ is a copy of $\beta N$, then $A$ is a union of finitely many anti-chains.

Example 3.3. There are two anti-chains $A', A'' \subseteq N$ such that $\overline{A'}$ and $\overline{A''}$ are copies of $\beta N$, but $\overline{A'} \cup \overline{A''}$ is not a copy of $\beta N$.

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2. Preliminaries

Bell's construction of the compactification $BN$ [1] follows.

Let $P = \{ f \in \omega^\omega : 0 \leq f(n) \leq n + 1 \text{ for all } n \in \omega \}$ and $N = \{ f|_n : f \in P, n \in \omega \}$.

Define $T = \{ \pi \in N^\omega : \text{dom } \pi(n) = n + 1 \text{ for all } n \in \omega \}$.

For every $s \in N$, let $C_s = \{ t \in N : t|_{\text{dom } s} = s \}$.

For every $\pi \in T$, let $C_{\pi(n)} = \{ n \in \omega \}$.

Let $B$ be the Boolean algebra, generated by $B' = \{ C_\pi : \pi \in T \}$.

in the power set of $N$.

Note that $\{ \{ s \} : s \in N \} \cup \{ C_s : s \in N \} \subseteq B$.

Denote by $BN$ the Stone space of $B$. We are identifying each $s \in N$ with the ultrafilter $\xi_s \in BN$ such that $\{ s \} \in \xi_s$. So $BN$ is a compactification of the countable discrete space $N$. We will denote points of $N$ as $f|_n, s, t$.

There is an order on $N$: $s \leq t$ if $t$ is an extension of $s$ for $s, t \in N$. We write $s < t$ if $s < t$ and $s \neq t$.

Recall that a subset $A$ of the ordered set $N$ is called an anti-chain if $A$ consists of incomparable points of $N$.

We call an anti-chain $A \subseteq N$ a strict anti-chain, if $\text{dom } x \neq \text{dom } y$ for all $x, y \in A, x \neq y$.

For $\pi \in T$ and $M \subseteq \omega$, define $C_{\pi|M} = \bigcup \{ C_{\pi(n)} : n \in M \}$.

It was proved in [5, Lemma 3.1] that $C_{\pi|M} \in B$.

The following results were proved in [5].

**Theorem 2.1.** The family

$$\hat{B} = \{ C_{\pi|M} \setminus \bigcup_{i < n} C_{\pi_i} : M \subseteq \omega, n \in \omega \text{ and } \{ \pi \} \cup \{ \pi_i : i < n \} \subseteq T \}$$

is a base for the topology of $BN$.

Theorem 2.1 follows from [5, Lemma 3.1 and Lemma 3.2] and from the fact that $\{ s \} \in \hat{B}$ for all $s \in N$.

The following result is [5, Lemma 3.5].

**Theorem 2.2.** Let $\{ s_i : i \in \omega \} \subseteq N$ be a strict anti-chain. If $x_i \in \overline{C_{s_i}}$ for each $i \in \omega$, then $\{ x_i : i \in \omega \}$ is homeomorphic to $\beta N$.

**Theorem 2.3.** Let $A = \{ s_i : i \in \omega \}$ be an infinite chain in $N$. Then $|A \setminus A| = 1$, i.e., $A$ is a convergent sequence in $BN$.

More results about $BN$ can be found in [4], [2], and [3].
3. **Main Results**

**Theorem 3.1.** If a set $A \subseteq N$ is such that $|A \setminus A| = 1$, then $A \setminus K$ is a chain for some finite $K \subseteq A$.

**Proof.** Let $A \subseteq N$ be such that $A \setminus A = \{x\}$. Then $|A| = \omega$ and $x \in BN \setminus N$. We claim that for all $s \in N$ the following is true:

\[
\begin{align*}
\text{If } A' \subseteq A \text{ is infinite, then either } C_s \cap A' \text{ is finite} \\
\text{or } (N \setminus C_s) \cap A' \text{ is finite.}
\end{align*}
\]

Otherwise, for some infinite $A' \subseteq A$, we have $C_s \cap A' \ni x$ and $(N \setminus C_s) \cap A'$ $\ni x$, but $C_s \cap N \setminus C_s = \emptyset$.

Denote $A_1 = \{s \in A : (N \setminus C_s) \cap A = \emptyset\}$. We will prove that $A_1$ is a finite set. Suppose $|A_1| = \omega$. We construct a strict anti-chain $\{s_i : i \in \omega\} = A_2 \subseteq A_1$.

Let $s_0 = s$ for some $s \in A_1$. Assume $\{s_i : i \leq n\}$ has been chosen. Since $s_i \in A_1$, by $(\ast)$, we have $|C_{s_i} \cap A| < \omega$, and therefore $|C_{s_i} \cap A| < \omega$.

Then $|\bigcup_{i \leq n} C_{s_i} \cap A| < \omega$, and we choose $s_{n+1} \in A_1 \setminus \bigcup_{i \leq n} C_{s_i}$ such that $\text{dom} s_{n+1} > \text{dom} s_i$ ($i \leq n$).

Note that $s_{n+1}$ is not an extension of $s_i$ for all $i \leq n$.

By construction, the set $A_2 = \{s_i : i < \omega\}$ is a strict anti-chain. By Theorem 2.2, $\mathcal{A}_2$ is homeomorphic to $\beta N$, but $\mathcal{A}_2 \setminus A_2 = \{x\}$, a contradiction. So $A_1$ is a finite set.

Denote $A = A \setminus A_1$.

We will prove that $\tilde{A}$ is a chain. Note that the following holds:

\[
\begin{align*}
|\tilde{A} \setminus C_s| < \omega \text{ for all } s \in \tilde{A}.
\end{align*}
\]

Let $s, s' \in \tilde{A}$ so that $s \neq s'$. If $s$ and $s'$ are not comparable, we obtain $C_s \cap C_{s'} = \emptyset$. Then $C_{s'} \cap \tilde{A} \subseteq \tilde{A} \setminus C_s$. But $C_{s'} \cap \tilde{A}$ is infinite and this contradicts $(\ast)$). So $\tilde{A}$ is a chain. \hfill $\square$

**Theorem 3.2.** If the closure $\mathcal{D}$ of a set $D \subseteq N$ is homeomorphic to $\beta N$, then $D$ is the union of finitely many anti-chains.

**Proof.** Let $D = \{s_n : n \in \omega\}$ be a subset of $N$ such that $\mathcal{D}$ is homeomorphic to $\beta N$. First, we prove that

\[
\begin{align*}
\ast \text{ there is a natural number } K \text{ such that the cardinality of every chain in } D \text{ is not more than } K.
\end{align*}
\]

For $s_n \in D$, denote

\[
P(s_n) = \{s_k \in D : s_k \leq s_n\}, \quad h(s_n) = |P(s_n)|.
\]
Note that each set \( P(s_n) \) is a chain and \( s_n = \max P(s_n) \). Note also that \( D \) contains no infinite chain. Indeed, an infinite chain is a converging sequence by Theorem 2.3 and this contradicts that \( D \) is homeomorphic to \( \beta N \). Thus, every chain in \( D \) is a subset of \( P(s_n) \) of some \( s_n \).

We prove that there is a number \( K \) such that \( h(s_n) \leq K \) for all \( s_n \in D \). Assume otherwise, i.e., that \( \{ h(s_n) : s_n \in D \} \) is unbounded. As a first step, we construct a sequence \( \{ s_{nk} : k \in \omega \} \) in the following way.

Let \( s_{n_k} \in D \) be such that \( h(s_{n_k}) = \min \{ h(s_n) : s_n \in D \} \). Assume we have chosen \( \{ s_{n_i} : i \leq k \} \). Choose \( s_{n_{k+1}} \) such that \( h(s_{n_{k+1}}) \geq \sum_{i=1}^{k} \text{dom } s_{n_i} + k \).

Consider the sequence \( \{ s_{n_k} : k \in \omega \} \). For all \( s_{n_k} (k \in \omega) \), define a finite set \( H(s_{n_k}) \) as follows:

\[
H(s_{n_k}) = P(s_{n_k}) \setminus \{ P(s_{n_i}) : i < k \}.
\]

Note that the family \( \{ H(s_{n_k}) : k \in \omega \} \) satisfies the following:

\[
|H(s_{n_k})| \geq k \text{ and } H(s_{n_k}) \cap H(s_{n_m}) = \emptyset \text{ if } s_{n_k} \neq s_{n_m}.
\]

For every set \( H(s_{n_k}) \), let \( \{ H_1(s_{n_k}), H_2(s_{n_k}) \} \) be a partition of \( H(s_{n_k}) \) such that

\[ (**) \text{ for every two elements } s \text{ and } t \text{ of one member of the partition satisfying } s \prec t, \text{ there is an element } r \text{ of the other member of the partition such that } s \prec r \prec t. \]

Denote \( H_i = \bigcup \{ H_i(s_{n_k}) : k \in \omega \} \) \((i = 1, 2)\). We have \( H_i \subseteq D \) \((i = 1, 2)\) and \( H_1 \cap H_2 = \emptyset \). We will prove that \( |H_1 \cap H_2| \neq \emptyset \) in order to get a contradiction with the condition that \( D \) is homeomorphic to \( \beta N \).

Assume that \( H_1 \cap H_2 = \emptyset \). There is a finite cover \( \lambda = \{ O_i : i = 1, \ldots, n \} \) of \( H_1 \) such that \( H_2 \cap (\bigcup_{i \in \lambda} O_i) = \emptyset \) where \( O_i \) are basic open sets of the form described in Theorem 2.1.

Let \( k \) be a natural number such that \( k > 2n + 1 \). Consider the sets \( H_1(s_{n_k}) \) and \( H_2(s_{n_k}) \) and note that \( |H_i(s_{n_k})| > n \) \((i = 1, 2)\).

We claim that if \( s, t \in H_1(s_{n_k}) \) are distinct, then there is no \( O_i \in \lambda \) such that \( s, t \in O_i \).

Assume otherwise and let \( s, t \in H_1(s_{n_k}) \) and \( O_i \in \lambda \) be such that \( s \prec t \) and \( s, t \in O_i \).

By \((**)\), there is \( r \in H_2(s_{n_k}) \) such that \( s \prec r \prec t \). Since \( O_i = C_{\pi|M} \setminus \bigcup_{j < t} C_{\pi_j} \), for some \( \{ \pi \} \cup \{ \pi_j : j < t \} \subseteq T \) and \( s \in O_i \), we have \( s \in C_{\pi|M} \). Since \( s \prec r \), we have \( r \in C_{\pi|M} \).

On the other hand, \( r \in H_2(s_{n_k}) \) implies \( r \notin O_i \). Since \( r \in C_{\pi|M} \), we obtain \( r \in \bigcup_{j < t} C_{\pi_j} \). But \( r \prec t \) and so \( t \in \bigcup_{j < t} C_{\pi_j} \); therefore, \( t \notin O_i \).
So we proved that distinct elements of the set \( H_1(s_{n_k}) \) are in distinct elements of the cover \( \lambda = \{O_i : i = 1, \ldots, n\} \), but this contradicts \( |H_1(s_{n_k})| > n \).

So we proved that \( \mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset \), but this contradicts the fact that \( \mathcal{D} \) is a copy of \( \beta N \). This contradiction proves that our assumption that \( \{h(s_n) : s_n \in D\} \) is unbounded is not true.

Note that since \( D \) contains no infinite chain, the following holds:

\[
\text{(m)} \quad \text{Every nonempty subset } D' \subset D \text{ has an element which is maximal in } D'.
\]

Here an element \( s \in D' \) is called \textit{maximal} in \( D' \) if there is no \( t \in D' \), such that \( s < t \).

We will construct the family of subsets of \( D \), \( \{D_i : i \in \omega\} \), as follows.

Let \( D_0 \subset D \) be the set of all elements of \( D \) which are maximal in \( D \).

Assume \( \{D_i : i \leq n\} \) has been constructed. If \( D \setminus \bigcup \{D_i : i \leq n\} \neq \emptyset \), let \( D_{n+1} \) be the set of all maximal elements of \( D \setminus \bigcup \{D_i : i \leq n\} \).

Note that \( D_n \) is an anti-chain for all \( n \) and \( D_n \cap D_m = \emptyset \) if \( n \neq m \).

Note also that by (m), \( D_n = \emptyset \) if and only if \( D \setminus \bigcup \{D_i : i < n\} = \emptyset \) and therefore, if \( D_n = \emptyset \) for some \( n \in \omega \), then \( D_m = \emptyset \) for all \( m > n \).

Let \( D_{n+1} \neq \emptyset \) for some \( n \). We will prove that

\[
\text{(***)} \quad \text{for every } s \in D_{n+1}, \text{ there is } q \in D_n \text{ such that } s < q.
\]

Let \( s \in D_{n+1} \). There is an element \( t \in D \) such that \( s < t \), otherwise \( s \in D_0 \). Moreover, if \( s < t \), then \( t \in \bigcup \{D_i : i \leq n\} \), because otherwise \( s \) is not maximal in \( D \setminus \bigcup \{D_i : i \leq n\} \), and \( s \notin D_{n+1} \). Let

\[
\ell = \max \{n : \text{there is } t \in D_n \text{ such that } s < t\}.
\]

Let us show that \( s \) is maximal in \( D \setminus \bigcup \{D_i : i \leq \ell\} \). Assume that the set \( A = \{r \in D \setminus \bigcup \{D_i : i \leq \ell\} : s < r\} \) is nonempty. By (m), \( A \) has a maximal element \( r_0 \). Then \( r_0 \) is maximal in \( D \setminus \bigcup \{D_i : i \leq \ell\} \), and therefore \( r_0 \in D_{\ell+1} \), but this contradicts (\( \ell \)). So \( s \) is maximal in \( D \setminus \bigcup \{D_i : i \leq \ell\} \), and therefore \( s \in D_{\ell+1} \). Then \( \ell = n \), because otherwise, since \( D_{\ell+1} \cap D_{n+1} = \emptyset \), we get \( s \notin D_{n+1} \). So (***') is proved.

Let \( K \) be a natural number such that the cardinality of every chain in \( D \) is not more than \( K \).

We will prove that \( D_K = \emptyset \) and therefore \( D = \bigcup \{D_i : i < K\} \). Assume otherwise; then \( D_i \neq \emptyset \) for all \( i = 0, \ldots, K \). By (**), for each \( i = 0, \ldots, K \), there exists \( s_i \in D_{K-i} \) such that \( \{s_i : i = 0, \ldots, K\} \) is a chain.

\[ \square \]

\textbf{Example 3.3.} There are two anti-chains \( A', A'' \subseteq N \) such that \( \overline{A'} \) and \( \overline{A''} \) are homeomorphic to \( \beta N \), but \( \overline{A'} \cup \overline{A''} \) is not homeomorphic to \( \beta N \).
Let $A = \{s_n : n \in \omega\}$ be an infinite strict anti-chain. Let $\lambda = \{A_m : m \in \omega\}$ be a partition of $A$ such that

1. $\text{dom}(s_t) < \text{dom}(s_{t'})$ for all $s_t \in A_m$, $s_{t'} \in A_{m'}$ and $m < m'$,
2. $|A_m| = m$.

Define $d_m = \max\{\text{dom} s_n : s_n \in A_m\}$.

We construct an anti-chain $A'$ as follows. For every $s_n \in A_m$, fix an element $s_n' \in N$ such that $\text{dom} s_n' = d_m$ and $s_n \leq s_n'$.

Let $A_m' = \{s_n' : s_n \in A_m\}$, $A' = \cup\{A_m' : m \in \omega\}$, and $\lambda' = \{A'_m : m \in \omega\}$.

Now we will define an anti-chain $A''$. For every $s'_n \in A'$, let $s''_n$ be a successor of $s'_n$, i.e., $s'_n < s''_n$ and dom $s''_n = \text{dom} s'_n + 1$. Let $A'' = \{s''_n : s'_n \in A'\}$. We prove that

1. $A'$ and $A''$ are homeomorphic to $\beta N$,
2. $\overline{A'} \cap \overline{A''} \neq \emptyset$ and therefore $\overline{A'} \cup \overline{A''}$ is not homeomorphic to $\beta N$.

For the strict anti-chain $A = \{s_n : n \in \omega\}$, there is an associated family $\{C_{s_n} : s_n \in A\}$ (see the definition of the Boolean algebra $B$). Note that $A' = \{s'_n : n \in \omega\}$ and $A'' = \{s''_n : n \in \omega\}$ are such that $s'_n$, $s''_n \in C_{s_n}$ for all $n \in \omega$. Therefore, by Theorem 2.2, $\overline{A'}$ and $\overline{A''}$ are homeomorphic to $\beta N$.

Let us regard $A' = \{s'_n : n \in \omega\}$.

Note that

(*) there is an ultrafilter $\xi'$ on $A'$ such that for every $F', F'' \in \xi'$ and for each natural number $k$, there is $A'_m \in \lambda'$ such that $|A'_m \cap F'| > k$.

Indeed, denote $\eta_k = \{E \subseteq A' : |E \cap A'_m| \leq k \text{ for all } m \in \omega\}$ for every $k \in \omega$ and $\eta = \cup \{\eta_k : k \in \omega\}$. Since $|A_m| = m$ for all $m \in \omega$, the family $\theta = \{(A' \setminus E : E \in \eta)\}$ has the finite intersection property. An ultrafilter $\xi'$ such that $\theta \subseteq \xi'$ is as required.

Let $\{x(\xi')\} = \cap\{\overline{F'} : F' \in \xi'\}$, then $x(\xi') \in \overline{A'}$.

We will show that $x(\xi') \in \overline{A''}$. Assume that there is a neighbourhood $Ox(\xi') = \overline{C_{\pi|M}} \setminus \bigcup_{i < k_0} \overline{C_{\pi}}$, of $x(\xi')$ for some $\{\pi\} \cup \{\pi_i : i < k_0\} \subseteq T$, such that $Ox(\xi') \cap A'' = \emptyset$.

Since $x(\xi') \in \overline{A'}$ and $\overline{A'}$ is homeomorphic to $\beta N$, $Ox(\xi') \cap A' \in \xi'$. Then, by (*), there is $A'_m \in \lambda'$ such that $|A'_m \cap Ox(\xi') \cap A'| = |A'_m \cap Ox(\xi')| > k_0$. By the construction of the sets $A'$ and $A''$, it follows that

(**) if $s'_n \in Ox(\xi') \cap A'_m$, then $s''_n \in C_{\pi|M}$, and hence $s''_n \in C_{\pi|M} \cap A''_m$.

Define $D = \{s''_n : s'_n \in Ox(\xi') \cap A'_m\}$. From (**) it follows that

$D \subseteq C_{\pi|M} \cap A'_m$ and $|D| = |Ox(\xi') \cap A'_m| > k_0$. 

From $Ox(\xi') \cap A'_m = \emptyset$ and (**), it follows that $D \subseteq \bigcup_{i < k_0} C_{\pi_i}$. Recall also that $\text{dom} s''_n = \text{dom} s'_n + 1 = d_m + 1$ and $s'_n < s''_n$ for all $s''_n \in A'_m$.

We claim that if $s''_n \in D \cap C_{\pi_i}$ for some $i < k_0$, then $s''_n \in C_{\pi_i}(d_m)$ and $\pi_i(d_m) = s''_n$. Indeed, if $s''_n \in C_{\pi_i}(l)$, then $l \leq d_m$. If $l < d_m$, then $s'_n \in C_{\pi_i}(l)$, but since $s''_n \in D$, we have $s'_n \in Ox(\xi') \cap A'_m$, and therefore $s'_n \notin \bigcup_{i < k_0} C_{\pi_i}$. It is a contradiction. So $l = d_m$.

From this it follows that if $s''_{n_1}, s''_{n_2} \in D$ are distinct and $s''_{n_1} \in C_{\pi_1}$ and $s''_{n_2} \in C_{\pi_2}$, then $i_1 \neq i_2$.

We get that $k_0 \geq |D|$, but $|D| > k_0$. It is a contradiction.

So $x(\xi') \in \overline{A'}$ and $\overline{A'} \cap \overline{A'} = \emptyset$.

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**References**


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